

More on Empirical Negation

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Abstract

Intuitionism can be seen as a verificationism restricted to mathematical discourse. An attempt to generalize intuitionism to empirical discourse presents various challenges. One of those concerns the logical and semantical behavior of what has been called ‘empirical negation’. An extension of intuitionistic logic with empirical negation was given by Michael De and a labelled tableaux system was there shown sound and complete. However, a Hilbert-style axiom system that is sound and complete was missing. In this paper we provide the missing axiom system which is shown sound and complete with respect to its intended semantics. Along the way we consider some further applications of empirical negation.

Keywords: Intuitionistic logic, empirical negation, completeness.

1 Introduction

A verificationism along the lines of Michael Dummett and Neil Tennant seeks to generalize a constructive interpretation of mathematical discourse to empirical discourse. A simplified strategy along these lines can be thought to be carried out as follows. First, take the famous Brouwer-Heyting-Kolmogorov (BHK) interpretation of the base (propositional) logical connectives, where provability is taken informally:

- (i) a conjunction $A \wedge B$ is provable iff A and B are;
- (ii) a disjunction $A \vee B$ is provable iff either A or B is;

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(iii) a conditional $A \rightarrow B$ is provable iff there is a method transforming any proof of A into a proof of B ;

(iv) \perp isn't provable.

As usual the negation $\neg A$ of A is defined by $A \rightarrow \perp$. Second, replace in those clauses 'provable' with 'verifiable', where a verification is intended to cover not only mathematical statements but empirical ones as well. Third and finally, provide a workable account of what verification amounts to. E.g. one such account might hold that a statement is verifiable iff it could be warranted, where warrants are proofs in the mathematical case and some other sort of evidence in the empirical case, depending on the domain of discourse.

Of course filling in the details of such an account is far from trivial. Besides meeting the onerous task of giving a plausible account of warrant or evidence, there are already significant worries concerning the logical language. Will \wedge , \vee , and \rightarrow and \perp suffice as propositional connectives for an empirical language? There are good reasons to think not, especially concerning negation. Dummett says:

Negation ... is highly problematic. In mathematics, given the meaning of "if ... then", it is trivial to explain "Not A" as meaning "If A, then $0 = 1$ "; by contrast, a satisfactory explanation of "not", as applied to empirical statements for which bivalence is not, in general, taken as holding, is very difficult to arrive at. Given that the sentential operators cannot be thought of as explained by means of the two-valued truth-tables, the possibility that the laws of classical logic will fail is evidently open: but it is far from evident that the correct logical laws will always be the intuitionistic ones. More generally, it is by no means easy to determine what should serve as the analogue, for empirical statements, of the notion of proof as it figures in intuitionist semantics for mathematical statements. [7, p.473]

One problem is that the "arrow-falsum" definition of negation is often too strong to serve as the negation for empirical statements. Suppose we attempt to express in our generalized intuitionistic language the fact that Goldbach's conjecture is not decided. We obtain the statement that says that any warrant for 'Goldbach's conjecture is decided' can be transformed into warrant for an absurdity (say ' $0=1$ '). Since there could be no proof of an absurdity, this statement says that Goldbach's conjecture is undecidable! But it might turn out in the future that someone prove or refute Goldbach's conjecture. So the fact that Goldbach's conjecture is not decided does not imply that it is undecidable, as our translation gives us. What we rather wished to say was merely that there is no sufficient evidence at present for the truth of the conjecture.

Couldn't we translate the original statement in a way that avoids this problem? For example, couldn't we translate 'Goldbach's conjecture is not decided' as 'If Goldbach's conjecture is decided *at present*, then $0=1$ '? Since the conjecture is not decided at present, the idea would be that we can (vacuously) turn any warrant or evidence for the antecedent of the conditional into evidence for an absurdity, for there is no evidence for the antecedent! The problem is that

if there were *any* evidence at all to a sufficient degree that Goldbach's conjecture is decided and if that evidence can be fallible (which is likely given that the statement is empirical), then it is not at all clear that that evidence can be transformed into a proof of ' $0=1$ '. Suppose in the future some reliable quantum computer has returned a very long but flawed proof of the conjecture. How exactly could that evidence be turned into evidence for ' $0=1$ '? That would depend on the nature of the mistake. If the conjecture is true and decidable and if from the mistake alone (with Peano arithmetic) one cannot infer that ' $0=1$ ', then it's hard to see how that evidence could be turned into evidence for ' $0=1$ '. If the conjecture is undecidable it's true, so again having evidence for or against it won't likely convert into evidence for an absurdity since the conjecture and its negation are both consistent with Peano arithmetic. If the conjecture is false (hence decidable), then there is still no reason to think that a flawed proof of a false conjecture will convert into evidence for an absurdity.³

There are cases that are more problematic than the one just mentioned, as they involve no mathematical content. Consider the purely empirical statement 'There are ten thousand leaves in my garden'. If that statement is false and yet after spending the entire day counting the leaves I (mistakenly) arrive at ten thousand, how precisely will my evidence be converted into evidence that $0=1$? One surely cannot use that evidence in a *derivation* in Peano arithmetic to ' $0=1$ '. We could choose a different absurdity since we are in a different domain of discourse, but exactly what would that absurdity be—one concerning features of my garden? Could any evidence concerning the wrong number of leaves in my garden be converted into evidence for this absurdity? This all seems unlikely. We are best to conclude that the fact that an empirical statement is false does not imply that *any* (fallible) evidence for it, whether that evidence be got now or in the future, can be converted into evidence for some given absurdity.⁴

We need, therefore, a *sui generis* empirical negation that is not definable from the standard (generalized) intuitionistic connectives. There have been some proposals as to how such a negation should behave logically and semantically in a verificationist setting. Two such accounts may be found in [4] and [6], with some general philosophical difficulties for such a project raised in [21]. In [4], intuitionistic logic is enriched by an empirical negation given the intuitive reading 'It is not the case that there is sufficient evidence at present that'. A tableaux system for the logic was there shown sound and complete, as no Hilbert-style axiom formulation could be found. We build on the work of [4] by

³ If the conjecture is false, then its negation is provable, so if we had a "proof" of the conjecture, could we not turn that into proof of an absurdity? Only if we also had the proof of the negation of the conjecture! Since we do not, it is not at all obvious that an alleged proof of the conjecture could *at any time* be converted into a proof of an absurdity.

⁴ Interestingly, empirical negation was discussed as far back as [10, p. 18] under the guise of 'factual' negation. Timothy Williamson [21] argues that sentences of the form '*A* will never be decided' are in fact inconsistent in a generalized intuitionistic language, even one with what he considers a plausible empirical negation. We will have more to say about this argument in section §4.

providing a Hilbert-style proof system for the logic which is shown sound and complete with respect to its intended semantics. We conclude by addressing some of the worries raised in [21].

2 Semantics and proof theory

After setting up the language, we first present the semantics, and then turn to the proof theory.

Definition 2.1 The language \mathcal{L} consists of a finite set $\{\sim, \wedge, \vee, \rightarrow\}$ of propositional connectives and a countable set \mathbf{Prop} of propositional variables which we denote by p, q , etc. Furthermore, we denote by \mathbf{Form} the set of formulas defined as usual in \mathcal{L} . We denote a formula of \mathcal{L} by A, B, C , etc. and a set of formulas of \mathcal{L} by Γ, Δ, Σ , etc.

2.1 Semantics

Definition 2.2 A model for the language is a quadruple $\langle W, g, \leq, V \rangle$, where W is a non-empty set (of states); $g \in W$ (the base state); \leq is a partial order on W with g being the least element; and $V : W \times \mathbf{Prop} \rightarrow \{0, 1\}$ an assignment of truth values to state-variable pairs with the condition that $V(w_1, p) = 1$ and $w_1 \leq w_2$ only if $V(w_2, p) = 1$ for all $p \in \mathbf{Prop}$ and all $w_1, w_2 \in W$.⁵ Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$
- $I(w, \sim A) = 1$ iff $I(g, A) = 0$
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$
- $I(w, A \rightarrow B) = 1$ iff for all $x \in W$: if $w \leq x$ and $I(x, A) = 1$ then $I(x, B) = 1$.

For a philosophical interpretation of the semantics, the reader is referred to [4].

Semantic consequence is now defined in terms of truth preservation at g : $\Sigma \models A$ iff for all models $\langle W, g, \leq, I \rangle$, $I(g, A) = 1$ if $I(g, B) = 1$ for all $B \in \Sigma$.

2.2 Proof Theory

Definition 2.3 The system \mathbf{IPC}^\sim consists of the following axiom schemata and rules of inference:

$$\begin{array}{ll}
 A \rightarrow (B \rightarrow A) & (\text{Ax1}) \\
 (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) & (\text{Ax2}) \\
 (A \wedge B) \rightarrow A & (\text{Ax3}) \\
 (A \wedge B) \rightarrow B & (\text{Ax4}) \\
 (C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow (A \wedge B))) & (\text{Ax5})
 \end{array}$$

⁵ Note here that in the semantics presented in [4], it is not assumed that the distinguished element g is a least element with respect to the partial ordering. This constraint doesn't affect the consequence relation but is needed for the completeness proof below.

$$\begin{array}{ll}
A \rightarrow (A \vee B) & (\text{Ax6}) \\
B \rightarrow (A \vee B) & (\text{Ax7}) \\
(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)) & (\text{Ax8}) \\
A \vee \sim A & (\text{Ax9}) \\
\sim A \rightarrow (\sim\sim A \rightarrow B) & (\text{Ax10}) \\
\frac{A \quad A \rightarrow B}{B} & (\text{MP}) \\
\frac{A \vee B}{\sim A \rightarrow B} & (\text{RP})
\end{array}$$

Following the usual convention, we define $A \leftrightarrow B$ as $(A \rightarrow B) \wedge (B \rightarrow A)$. Finally, we write $\Gamma \vdash A$ if there is a sequence of formulas B_1, \dots, B_n, A , $n \geq 0$, such that every formula in the sequence B_1, \dots, B_n, A either (i) belongs to Γ ; (ii) is an axiom of \mathbf{IPC}^\sim ; (iii) is obtained by (MP) or (RP) from formulas preceding it in sequence.

Remark 2.4 We will refer to the subsystem of \mathbf{IPC}^\sim which consists of axiom schemata from (Ax1) to (Ax8) and a rule of inference (MP) as \mathbf{IPC}^+ . Note that the deduction theorem does not hold with respect to \rightarrow in this system, as observed by De. However, we do have a deduction theorem in a slightly different form, given below as Theorem 2.11.

Our goal now is to prove a variant of the deduction theorem. For this purpose, we begin with some preparations.

Fact 2.5 *The following formulas are provable in \mathbf{IPC}^+ and thus in \mathbf{IPC}^\sim .*

$$\begin{array}{ll}
A \rightarrow A & (1) \\
(A \vee B) \rightarrow (B \vee A) & (2) \\
(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) & (3) \\
(A \vee B) \rightarrow ((B \rightarrow C) \rightarrow (A \vee C)) & (4) \\
(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C) & (5) \\
(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C) & (6)
\end{array}$$

Lemma 2.6 *The following formulas are provable in \mathbf{IPC}^\sim .*

$$\begin{array}{ll}
(A \rightarrow \sim A) \rightarrow \sim A & (7) \\
(\sim A \rightarrow A) \rightarrow A & (8) \\
\sim\sim A \rightarrow A & (9) \\
(\sim A \rightarrow B) \rightarrow (A \vee B) & (10) \\
(\sim\sim A \rightarrow B) \leftrightarrow (\sim A \vee B) & (11)
\end{array}$$

Proof. (7) can be derived by making use of (Ax8), (1), (3) and (Ax9); and (8) is similarly derived. (9) is proved by (Ax9), (2) and (RP). (10) is also easy to derive by combining (4) and (Ax9). For (11), the left-to-right direction is (10). For the other way around, we only need to apply (Ax8) to (Ax1) and (Ax10). \square

Now, we can prove one direction of the deduction theorem.

Proposition 2.7 For $\Gamma \cup \{A, B\} \subseteq \text{Form}$, if $\Gamma, A \vdash B$ then $\Gamma \vdash \sim\sim A \rightarrow B$.

Proof. By the induction on the length n of the proof of $\Gamma, A \vdash B$. If $n = 1$, then we have the following three cases.

- If B is one of the axioms of IPC^\sim , then we have $\vdash B$. Therefore, by (Ax1), we obtain $\vdash \sim\sim A \rightarrow B$ which implies the desired result.
- If $B \in \Gamma$, then we have $\Gamma \vdash B$, and thus we obtain the desired result by (Ax1).
- If $B = A$, then by (9), we have $\sim\sim A \rightarrow B$ which implies the desired result.

For $n > 1$, then there are two additional cases to be considered.

- If B is obtained by applying (MP), then we will have $\Gamma, A \vdash C$ and $\Gamma, A \vdash C \rightarrow B$ lengths of the proof of which are less than n . Thus, by induction hypothesis, we have $\Gamma \vdash \sim\sim A \rightarrow C$ and $\Gamma \vdash \sim\sim A \rightarrow (C \rightarrow B)$, and by (Ax2) and (MP), we obtain $\Gamma \vdash \sim\sim A \rightarrow B$ as desired.
- If B is obtained by applying (RP), then $B = \sim C \rightarrow D$ and we will have $\Gamma, A \vdash C \vee D$ length of the proof of which is less than n . Thus, by induction hypothesis, we have $\Gamma \vdash \sim\sim A \rightarrow (C \vee D)$. By (11) and (RP), we have $\Gamma \vdash \sim C \rightarrow (\sim A \vee D)$. Another application of (11) gives us $\Gamma \vdash \sim C \rightarrow (\sim\sim A \rightarrow D)$ and thus by exchange, we obtain $\Gamma \vdash \sim\sim A \rightarrow (\sim C \rightarrow D)$, i.e. $\Gamma \vdash \sim\sim A \rightarrow B$ as desired.

This completes the proof. \square

For the purpose of proving the other direction of the deduction theorem, we need another lemma.

Lemma 2.8 The following formulas are provable and rules are derivable in IPC^\sim .

$$\frac{A \rightarrow B}{\sim B \rightarrow \sim A} \quad (\text{RC})$$

$$\sim(A \rightarrow A) \rightarrow B \quad (12)$$

$$\sim\sim(A \rightarrow A) \quad (13)$$

$$\frac{A}{\sim\sim A} \quad (\text{RD})$$

Proof. For (RC), assume $A \rightarrow B$. Then by making use of (4) and (Ax9), we have $\sim A \vee B$ which is equivalent to $B \vee \sim A$ by (2). Thus by applying (RP), we obtain $\sim B \rightarrow \sim A$. For (12), note first that $\sim B \rightarrow (A \rightarrow A)$ is derivable by (1) and (Ax1). Then by (RC), we obtain $\sim(A \rightarrow A) \rightarrow \sim\sim B$. This together with (9) implies the desired result. Then, (13) follows by (12), taking $\sim\sim(A \rightarrow A)$ in place of B , and (7). For (RD), assume A . Then by (Ax1), we obtain $(A \rightarrow A) \rightarrow A$. By applying (RC) twice, we get $\sim\sim(A \rightarrow A) \rightarrow \sim\sim A$. The desired result follows by this and (13). \square

Remark 2.9 Note that $A \rightarrow \sim\sim A$, a stronger form of (RD), is *not* derivable, although we have the other way around (cf. (9)). In this sense the behavior

of double empirical negation is dual to the behavior of double intuitionistic negation. Note also that based on (12), we may define the bottom element \perp by $\sim(A \rightarrow A)$, instead of taking it as primitive. In view of (Ax10) and (5), $\sim A \wedge \sim\sim A$ also serves as a suitable definition of \perp . We can thus define intuitionistic negation \neg by the usual “arrow-falsum” definition: $\neg A := A \rightarrow \perp$.

Proposition 2.10 *For $\Gamma \cup \{A, B\} \subseteq \text{Form}$, if $\Gamma \vdash \sim\sim A \rightarrow B$ then $\Gamma, A \vdash B$.*

Proof. By the assumption $\Gamma \vdash \sim\sim A \rightarrow B$, we have $\Gamma, \sim\sim A \vdash B$ by (MP). Moreover, we have $\Gamma, A \vdash \sim\sim A$ by (RD). Thus, we obtain the desired result. \square

By combining Propositions 2.7 and 2.10, we obtain the following theorem.

Theorem 2.11 *For $\Gamma \cup \{A, B\} \subseteq \text{Form}$, $\Gamma, A \vdash B$ iff $\Gamma \vdash \sim\sim A \rightarrow B$.*

Corollary 2.12 *For $\Gamma \cup \{A, B\} \subseteq \text{Form}$, we have $\Gamma, A \vdash B$ iff $\Gamma \vdash \sim A \vee B$.*

Proof. Immediate in view of the above result and (11). \square

Remark 2.13 The deduction theorem formulated in terms of \sim and \vee is already discussed by De in [4, p.63] in which he takes the formula $\sim A \vee \sim\sim B$ instead of $\sim A \vee B$. However, these formulas are equivalent in the sense that both $\sim A \vee B \vdash \sim A \vee \sim\sim B$ and $\sim A \vee \sim\sim B \vdash \sim A \vee B$ hold.⁶ Thus, the deduction theorem discussed in [4] is equivalent to the one presented here although the version here is slightly simpler.

The following proposition shows that the de Morgan laws with respect to empirical negation are fully provable in IPC^\sim .

Proposition 2.14 *The following formulas are provable in IPC^\sim .*

$$\sim(A \vee B) \rightarrow (\sim A \wedge \sim B) \quad (14)$$

$$(\sim A \vee \sim B) \rightarrow \sim(A \wedge B) \quad (15)$$

$$\sim(A \wedge B) \rightarrow (\sim A \vee \sim B) \quad (16)$$

$$(\sim A \wedge \sim B) \rightarrow \sim(A \vee B) \quad (17)$$

Proof. (14) and (15) are essentially by (Ax3), (Ax4) and (Ax6), (Ax7) respectively together with (RC). For (16), it runs as follows.

$$\begin{array}{ll} 1 & \sim(\sim A \vee \sim B) \rightarrow (\sim\sim A \wedge \sim\sim B) & [(14)] \\ 2 & \sim(\sim A \vee \sim B) \rightarrow (A \wedge B) & [1, (9)] \\ 3 & \sim(A \wedge B) \rightarrow \sim\sim(\sim A \vee \sim B) & [2, (RC)] \\ 4 & \sim(A \wedge B) \rightarrow (\sim A \vee \sim B) & [3, (9)] \end{array}$$

Finally, for (17), it suffices to prove $\sim\sim(A \vee B) \rightarrow \sim(\sim A \wedge \sim B)$, and in view of (11), it suffices to prove $\sim(A \vee B) \vee \sim(\sim A \wedge \sim B)$. But then by (15) and (16), the formula is equivalent to $\sim((A \vee B) \wedge \sim A) \vee \sim\sim B$. And in view of the deduction theorem, it suffices to show $(A \vee B) \wedge \sim A \vdash \sim\sim B$. And this can

⁶ For the latter, something stronger holds, i.e. $(\sim A \vee \sim\sim B) \rightarrow (\sim A \vee B)$. The former is derivable in view of (RD) and Proposition 3.4 which is proved later. Note that, semantically, only $\sim A \vee \sim\sim B$ defines material implication at the base state in the sense that $\sim A \vee \sim\sim B$ is true at an arbitrary state iff A is false or B is true at the base state.

be proved as follows. Assume $(A \vee B) \wedge \sim A$. Then by applying (RP) to the first conjunct, we obtain $\sim A \rightarrow B$. Since we have $\sim A$ as our second conjunct of the assumption, we obtain B by (MP). Finally by applying (RD), we obtain the desired result. This completes the proof. \square

Aside from having proved some important validities, it is helpful to consider some invalidities. The following proposition provides a list of such notables.

Proposition 2.15 *The following formulas are invalid in \mathbf{IPC}^\sim .*

$$\begin{array}{ll} (A \wedge \sim B) \rightarrow \sim(A \rightarrow B) & (\sim A \rightarrow \sim B) \rightarrow (B \rightarrow A) \\ (A \wedge \sim A) \rightarrow B & \sim(A \rightarrow B) \rightarrow (A \wedge \sim B) \\ \neg(A \wedge \sim A) & \sim A \rightarrow \neg A \\ (A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A) & \end{array}$$

Note, however, that the rule-forms of all formulas in the left-hand-column are valid.

Proof. To see the invalidity of the above formulas, note that the invalid $(A \wedge \sim A) \rightarrow B$ follows from the others. The details are left to the interested reader.

As for the validity of the rule-forms, let us briefly sketch their proofs. Since the last one is (RC) of Lemma 2.8, we deal with the other three rules. For the second one, assume A and $\sim A$. Then, by applying (RD) to A , we get $\sim \sim A$. Therefore, this together with the other assumption $\sim A$ and (Ax10) gives us B as desired. Proof of third rule is exactly parallel. For the first, assume $A \wedge \sim B$ and $A \rightarrow B$. Then we obtain B and $\sim B$ by (Ax3) and (MP). So, by the rule we just proved, we can in particular derive $\sim(A \rightarrow B)$. That is, we have $A \wedge \sim B, A \rightarrow B \vdash \sim(A \rightarrow B)$. Then, by the variant of deduction theorem (Theorem 2.11), we get $A \wedge \sim B \vdash \sim \sim(A \rightarrow B) \rightarrow \sim(A \rightarrow B)$. Thus by (8), we obtain $A \wedge \sim B \vdash \sim(A \rightarrow B)$ as desired. \square

We have seen that the rule (RP) plays an important role. One may wonder whether this rule is derivable or not.⁷ The following proposition shows that the rule is independent of other axioms and rule (MP).

Proposition 2.16 *The rule (RP) is independent of the other axioms and rule (MP).*

Proof. Consider the following truth tables which characterize the logic \mathbf{P}^1 introduced by Antonio Sette in [18]:

\wedge	1	i	0	\vee	1	i	0	\rightarrow	1	i	0	\sim	1	0
1	1	1	0	1	1	1	1	1	1	1	0	1	0	0
i	1	1	0	i	1	1	1	i	1	1	0	i	1	1
0	0	0	0	0	1	1	0	0	1	1	1	0	0	1

Note here that designated values are 1 and i . It is straightforward to verify that the above truth tables validate all the axiom schemata of \mathbf{IPC}^\sim , and that (MP) preserves designationhood. However, if we assign the values i and 0 to

⁷ We thank a referee for asking us to clarify this point.

A and B of (RP), then we see that designated values are not preserved. Thus, we obtain the desired result. \square

Remark 2.17 Before turning to further results on \mathbf{IPC}^\sim , let us briefly mention some related systems. There are at least two closely related systems in the literature. One of them is the \mathbf{TCC}_ω of A. B. Gordienko which is introduced and studied in [9]. This extends Richard Sylvan’s \mathbf{CC}_ω , studied in detail in [19] which is motivated by the fact that \mathbf{C}_ω of Newton da Costa (cf. [3]) lacks intersubstitutivity of provable equivalents:

$$\frac{A \leftrightarrow B}{D(A) \leftrightarrow D(B)}$$

where $D(A)$ is some wff containing A and $D(B)$ results from $D(A)$ by replacing one (derivatively, zero or more) occurrence of A by B . Now, \mathbf{C}_ω is obtained by adding (Ax9) and (9) to \mathbf{IPC}^+ . Then, \mathbf{CC}_ω is obtained by adding (RC) to \mathbf{C}_ω , and \mathbf{TCC}_ω extends \mathbf{CC}_ω by adding (Ax10). This shows that \mathbf{TCC}_ω is a subsystem of \mathbf{IPC}^\sim , and since (RP) is not valid in \mathbf{TCC}_ω , it is a strict subsystem of \mathbf{IPC}^\sim .

The other closely related system is WECQ of Thomas Ferguson considered in [8]. This extends Graham Priest’s da Costa Logic \mathbf{daC} introduced and studied in [15]. Although the proof theory of \mathbf{daC} is given in terms of natural deduction by Priest, we may easily observe that the subsystem of \mathbf{IPC}^\sim obtained by eliminating (Ax10) is *weakly* complete with respect to the semantics developed in [15].⁸ Then one of the observations of Ferguson shows that the addition of (Ax10) in the proof theory corresponds to the addition of the following condition on R dubbed “backwards convergence”: if tRv and uRv , then there exists an w such that wRt and wRu . Note here that besides backward convergence, the relation R satisfies the usual conditions deployed in Kripke semantics for intuitionistic logic.⁹

3 Soundness and completeness

We now proceed to the proof of soundness and completeness. Our proof follows an idea employed in [16]. Since the system we deal with is not a relevant logic, the proof is rather simplified compared to that for \mathbf{B}^+ and its related systems.

3.1 Soundness

Theorem 3.1 For $\Gamma \cup \{A\} \subseteq \text{Form}$, if $\Gamma \vdash A$ then $\Gamma \models A$.

Proof. By induction on the length of the proof, as usual. \square

3.2 Preliminaries and key notions for completeness

As a preliminary for the completeness proof, we prove some rules that will be used in the following, and also some rules that emphasize some differences from

⁸ There is another axiomatization in terms of Hilbert-style calculus in [2], although the consequence relation is defined in an idiosyncratic way.

⁹ More on the relation between these systems can be found in [14].

\mathbf{B}^+ .

Lemma 3.2 *The following rules are derivable in \mathbf{IPC}^\sim .*

$$\begin{array}{c} \frac{C \rightarrow D}{(A \rightarrow C) \rightarrow (A \rightarrow D)} \quad \text{(Prefixing)} \\ \frac{A \rightarrow B}{(B \rightarrow C) \rightarrow (A \rightarrow C)} \quad \text{(Suffixing)} \\ \frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C} \quad \text{(Transitivity)} \\ \frac{C \vee (A \rightarrow B) \quad C \vee A}{C \vee B} \quad \text{(D-MP)} \\ \frac{C \vee (A \vee B)}{C \vee (\sim A \rightarrow B)} \quad \text{(D-RP)} \end{array}$$

Proof. We only deal with the last two rules as the others are obvious by the fact that \mathbf{IPC}^\sim extends \mathbf{IPC}^+ .

For (D-MP): Assume $C \vee (A \rightarrow B)$ and $C \vee A$. Then, by applying (RP), we obtain $\sim C \rightarrow (A \rightarrow B)$ and $\sim C \rightarrow A$ respectively. Combining these with (Ax2) will give us $\sim C \rightarrow B$. Finally, we apply (10) to obtain the desired result. For (D-RP): Assume $C \vee (A \vee B)$. Then, by applying (RP), we obtain $\sim(C \vee A) \rightarrow B$. Therefore, we obtain $(\sim C \wedge \sim A) \rightarrow B$ by using (17) which is equivalent to $\sim C \rightarrow (\sim A \rightarrow B)$. Finally, we apply (10) to obtain the desired result. \square

Remark 3.3 (D-MP) and (D-RP) show that disjunctive forms of (MP) and (RP) can be proved in \mathbf{IPC}^\sim , and thus we do not have to assume them as rules of inference as in \mathbf{B}^+ .

Since we have the deduction theorem, we can prove the following metatheorem in a rather simple manner.

Proposition 3.4 *If $A \vdash C$ and $B \vdash C$, then $A \vee B \vdash C$.*

Proof. Assume $A \vdash C$ and $B \vdash C$. Then, by Theorem 2.11, we obtain $\vdash \sim\sim A \rightarrow C$ and $\vdash \sim\sim B \rightarrow C$ respectively. By (Ax7), we get $\vdash (\sim\sim A \vee \sim\sim B) \rightarrow C$, and therefore $\vdash \sim\sim(A \vee B) \rightarrow C$ by (14) and (17). Finally, we obtain the desired result by another application of Theorem 2.11. \square

We now state some definitions that will play important roles in the proof.

- (i) $\Sigma \vdash_\pi A$ iff $\Sigma \cup \Pi \vdash A$.
- (ii) Σ is a Π -theory iff:
 - (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$
 - (b) if $\vdash_\pi A \rightarrow B$ then (if $A \in \Sigma$ then $B \in \Sigma$).
- (iii) Σ is *prime* iff (if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$).
- (iv) $\Sigma \vdash_\pi \Delta$ iff for some $D_1, \dots, D_n \in \Delta$, $\Sigma \vdash_\pi D_1 \vee \dots \vee D_n$.
- (v) $\vdash_\pi \Sigma \rightarrow \Delta$ iff for some $C_1, \dots, C_n \in \Sigma$ and $D_1, \dots, D_m \in \Delta$:

$$\vdash_\pi C_1 \wedge \dots \wedge C_n \rightarrow D_1 \vee \dots \vee D_m.$$

- (vi) Σ is Π -deductively closed iff (if $\Sigma \vdash_{\pi} A$ then $A \in \Sigma$).
- (vii) Let **Form** be the set of formulas. Then, $\langle \Sigma, \Delta \rangle$ is a Π -partition iff:
 - (a) $\Sigma \cup \Delta = \mathbf{Form}$
 - (b) $\not\vdash_{\pi} \Sigma \rightarrow \Delta$
- (viii) Σ is *non-trivial* iff $A \notin \Sigma$ for some formula A .

In all of the above, if Π is \emptyset , then the prefix ‘ Π -’ will simply be omitted.

Remark 3.5 One point of departure from [16] is that we do not need the definition of the set Π_{\rightarrow} which is defined as the set of all members of Π of the form $A \rightarrow B$ where Π is a set of sentences. Moreover, we added a definition of non-triviality that is not required in the proof of [16] as there is no bottom element in \mathbf{B}^+ . The definition is necessary for our proof since a bottom element is available in \mathbf{IPC}^{\sim} .

We here note the following useful fact which is not the case in \mathbf{B}^+ as it relies on (Ax1).

Lemma 3.6 *If Γ is Π -theory, then $\Pi \subseteq \Gamma$.*

Proof. Take $A \in \Pi$. Then, we have $\Pi \vdash A$. Now, take any $C \in \Gamma$. Then, by (Ax1), we obtain $\Pi \vdash C \rightarrow A$, i.e. $\vdash_{\pi} C \rightarrow A$. Thus, combining this together with $C \in \Gamma$ and the assumption that Γ is Π -theory, we conclude that $A \in \Gamma$. \square

3.3 Extension lemmas

We now prove a number of lemmas. The first group concerns extensions of sets with various properties.

Lemma 3.7 *If $\langle \Sigma, \Delta \rangle$ is a Π -partition then Σ is a prime Π -theory.*

Proof. We need to prove the following three facts:

- (i) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$.
- (ii) if $\vdash_{\pi} A \rightarrow B$ then (if $A \in \Sigma$ then $B \in \Sigma$).
- (iii) if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$.

For (1): Assume $A, B \in \Sigma$ and $A \wedge B \notin \Sigma$. Then since $\Sigma \cup \Delta = \mathbf{Form}$, we have $A \wedge B \in \Delta$. This immediately implies $\vdash_{\pi} \Sigma \rightarrow \Delta$ which is a contradiction in view of $\not\vdash_{\pi} \Sigma \rightarrow \Delta$.

For (2): Assume $\vdash_{\pi} A \rightarrow B$ and $A \in \Sigma$ and $B \notin \Sigma$. Then since $\Sigma \cup \Delta = \mathbf{Form}$, we have $B \in \Delta$. This means $\vdash_{\pi} \Sigma \rightarrow \Delta$ which is a contradiction in view of $\not\vdash_{\pi} \Sigma \rightarrow \Delta$.

For (3): Assume $A \vee B \in \Sigma$ and $A \notin \Sigma$ and $B \notin \Sigma$. Then, since $\Sigma \cup \Delta = \mathbf{Form}$, we have $A, B \in \Delta$. This immediately implies $\vdash_{\pi} \Sigma \rightarrow \Delta$ which is a contradiction in view of $\not\vdash_{\pi} \Sigma \rightarrow \Delta$. \square

Lemma 3.8 *If $\not\vdash_{\pi} \Sigma \rightarrow \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a Π -partition.*

Proof. The details are spelled out in the appendix. \square

Corollary 3.9 *Let Σ be a Π -theory, Δ be closed under disjunction, and $\Sigma \cap \Delta = \emptyset$. Then there is $\Sigma' \supseteq \Sigma$ such that $\Sigma' \cap \Delta = \emptyset$ and Σ' is a prime Π -theory.*

Proof. First, it follows that $\not\vdash_{\pi} \Sigma \rightarrow \Delta$. For otherwise there would be some $C_1, \dots, C_n \in \Sigma$ and $D_1, \dots, D_m \in \Delta$:

$$\vdash_{\pi} C_1 \wedge \dots \wedge C_n \rightarrow D_1 \vee \dots \vee D_m.$$

Then, since Σ be a Π -theory, Σ is closed under conjunction, so $C_1 \wedge \dots \wedge C_n \in \Sigma$. Moreover, if Σ be a Π -theory and $C_1 \wedge \dots \wedge C_n \in \Sigma$ and $\vdash_{\pi} C_1 \wedge \dots \wedge C_n \rightarrow D_1 \vee \dots \vee D_m$, then it follows that $D_1 \vee \dots \vee D_m \in \Sigma$. On the other hand, Δ be closed under disjunction so $D_1 \vee \dots \vee D_m \in \Delta$. By combining these, we obtain $D_1 \vee \dots \vee D_m \in \Sigma \cap \Delta$ which is a contradiction in view of $\Sigma \cap \Delta = \emptyset$.

Now since we have $\not\vdash_{\pi} \Sigma \rightarrow \Delta$, we obtain, by the previous lemma, that there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a Π -partition. And by Lemma 3.7, it follows that Σ' is a prime Π -theory. It also follows that $\Sigma' \cap \Delta = \emptyset$, for otherwise we will have a formula $A_0 \in \Sigma' \cap \Delta \subseteq \Sigma' \cap \Delta'$. This will immediately imply that $\vdash_{\pi} \Sigma' \rightarrow \Delta'$ which cannot be the case in view of the fact that $\langle \Sigma', \Delta' \rangle$ is a Π -partition. \square

Lemma 3.10 *If $\Sigma \not\vdash \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a partition, and Σ' is deductively closed.*

Proof. Similar to the proof of Lemma 3.8. The details are spelled out in the appendix. \square

Corollary 3.11 *If $\Sigma \not\vdash A$ then there are $\Pi \supseteq \Sigma$ such that $A \notin \Pi$, Π is a prime Π -theory and is Π -deductively closed.*

Proof. Let Δ be $\{A\}$ and Σ' be Π in Lemma 3.10. Then by the lemma, we obtain a Π such that $\Pi \supseteq \Sigma$, $\langle \Pi, \Delta' \rangle$ is a partition, and Π is deductively closed. By Lemma 3.7, it follows that Π is a prime theory. Furthermore, $A \notin \Pi$ is obvious by the construction of Π , and Π is Π -deductively closed since if $\Pi \vdash_{\pi} A$ then $\Pi \vdash A$. Since Π is deductively closed, it follows that $A \in \Pi$, as desired. It remains to be shown that Π is a Π -theory. Suppose that $\vdash_{\pi} C \rightarrow D$ and $C \in \Pi$. Then clearly $\Pi \vdash D$, and hence $D \in \Pi$ by deductive closure. \square

3.4 Counter-example lemma

The second lemma establishes that there are certain theories with properties that are crucial in the recursion case for \rightarrow in the proof of the main theorem.

Lemma 3.12 *If Π is a prime Π -theory that is Π -deductively closed and $A \rightarrow B \notin \Pi$, then there is a prime Π -theory Γ , such that $A \in \Gamma$ and $B \notin \Gamma$.*

Proof. Let $\Sigma = \{C : A \rightarrow C \in \Pi\}$. Then Σ is a Π -theory. For suppose that $C_1, C_2 \in \Sigma$. Then $A \rightarrow C_1, A \rightarrow C_2 \in \Pi$. Thus $\vdash_{\pi} A \rightarrow (C_1 \wedge C_2)$ by (Ax5), so $A \rightarrow (C_1 \wedge C_2) \in \Pi$ since Π is Π -deductively closed, and thus $C_1 \wedge C_2 \in \Sigma$ by the definition of Σ . Now suppose that $\vdash_{\pi} C \rightarrow D$ and $C \in \Sigma$. Then $A \rightarrow C \in \Pi$ and so $\vdash_{\pi} A \rightarrow C$. By (Transitivity) we have $\vdash_{\pi} A \rightarrow D$. Since Π is Π -deductively closed, we have $A \rightarrow D \in \Pi$, and by the definition of Σ , we obtain $D \in \Sigma$ as desired.

Clearly $A \in \Sigma$ and $B \vee \dots \vee B \notin \Sigma$. Based on this, let Δ be the closure of $\{B\}$ under disjunction. Then, $\Sigma \cap \Delta = \emptyset$. The result then follows from Corollary 3.9. \square

3.5 Completeness

We are finally ready to prove the completeness.

Theorem 3.13 *For $\Gamma \cup \{A\} \subseteq \text{Form}$, if $\Gamma \models A$ then $\Gamma \vdash A$.*

Proof. We prove the contrapositive. Suppose that $\Gamma \not\vdash A$. Then, by the above lemma, there is a $\Pi \supseteq \Gamma$ such that Π is a prime theory and $A \notin \Pi$. Define the interpretation $\mathfrak{A} = \langle \Pi, X, \leq, I \rangle$, where $X = \{\Delta : \Delta \text{ is a non-trivial prime } \Pi\text{-theory}\}$, $\Delta \leq \Sigma$ iff $\Delta \subseteq \Sigma$ and I is defined thus. For every state Σ and propositional parameter p :

$$I(\Sigma, p) = 1 \text{ iff } p \in \Sigma$$

We show that this condition holds for any arbitrary formula B :

$$I(\Sigma, B) = 1 \text{ iff } B \in \Sigma \tag{*}$$

It then follows that \mathfrak{A} is a counter-model for the inference, and hence that $\Gamma \not\models A$. The proof of (*) is by induction on the complexity of B .

Disjunction:

$$\begin{aligned} I(\Sigma, C \vee D) = 1 &\text{ iff } I(\Sigma, C) = 1 \text{ or } (I(\Sigma, D) = 1 \\ &\text{ iff } C \in \Sigma \text{ or } D \in \Sigma && \text{IH} \\ &\text{ iff } C \vee D \in \Sigma && \Sigma \text{ is a prime theory} \end{aligned}$$

Conjunction:

$$\begin{aligned} I(\Sigma, C \wedge D) = 1 &\text{ iff } I(\Sigma, C) = 1 \text{ and } (I(\Sigma, D) = 1 \\ &\text{ iff } C \in \Sigma \text{ and } D \in \Sigma && \text{IH} \\ &\text{ iff } C \wedge D \in \Sigma && \Sigma \text{ is a theory} \end{aligned}$$

Negation:

$$\begin{aligned} I(\Sigma, \sim C) = 1 &\text{ iff } I(\Pi, C) \neq 1 \\ &\text{ iff } C \notin \Pi && \text{IH} \\ &\text{ iff } \sim C \in \Sigma \end{aligned}$$

For the last equivalence, assume $C \notin \Pi$ and $\sim C \notin \Sigma$. Then, by the latter and $\Pi \subseteq \Sigma$, we obtain $\sim C \notin \Pi$. This together with $C \notin \Pi$ and the primeness of Π implies $C \vee \sim C \notin \Pi$, and thus $\Pi \not\vdash C \vee \sim C$, which is a contradiction. For the other way around, assume $\sim C \in \Sigma$ and $C \in \Pi$. Then, by the latter and $\Pi \subseteq \Sigma$, we obtain $C \in \Sigma$. Therefore, together with $\sim C \in \Sigma$, it follows that $\Sigma \vdash C \wedge \sim C$ and thus $\Sigma \vdash B$ for any B . This contradicts that Σ is non-trivial.

Conditional:

$$\begin{aligned}
I(\Sigma, C \rightarrow D) = 1 \text{ iff for all } \Delta \text{ s.t. } \Sigma \subseteq \Delta, \text{ if } I(\Delta, C) = 1 \text{ then } I(\Delta, D) = 1 \\
\text{iff for all } \Delta \text{ s.t. } \Sigma \subseteq \Delta, \text{ if } C \in \Delta \text{ then } D \in \Delta & \quad \text{IH} \\
\text{iff } C \rightarrow D \in \Sigma
\end{aligned}$$

For the last equivalence, assume $C \rightarrow D \in \Sigma$ and $C \in \Delta$ for any Δ s.t. $\Sigma \subseteq \Delta$. Then by $\Sigma \subseteq \Delta$ and $C \rightarrow D \in \Sigma$, we obtain $C \rightarrow D \in \Delta$. Therefore, we have $\Delta \vdash C \rightarrow D$ and $\Delta \vdash C \rightarrow D$, so by (MP), we obtain $\Delta \vdash D$, i.e. $D \in \Delta$, as desired. On the other hand, suppose $C \rightarrow D \notin \Sigma$. Then by the lemma above, there is a $\Sigma' \supseteq \Sigma$ such that $C \in \Sigma'$, $D \notin \Sigma'$ and Σ' is a prime Π -theory. And if Σ' is a prime Π -theory, then it follows that $\Pi \subseteq \Sigma'$. Furthermore, non-triviality of Σ' is obvious by $D \notin \Sigma'$. Thus, we obtain the desired result. \square

4 Some reflections

We now consider some related issues. First, we deal with a variant of the semantics we presented in §2. Second, we respond to some remarks of Williamson on empirical negation.

4.1 A “many distinguished states” semantics

Instead of taking a single base state as representing our current state of evidence, we may also think of having a set of base states representing ways our current state of evidence might be.¹⁰ In practice we can't always be certain about what our evidential state is because e.g. whether something counts as evidence or exactly what the evidence is might in principle be indeterminate. All we can do is rule out certain states as ours if they make false statements we know are supported by our current evidence. Since ruling out certain states as our own won't always secure a unique state, we might be more cautious to consider several states that are, for all we know or even could know, our current evidential state. This leads to the following modification of our semantics as follows.

Definition 4.1 A model for the language is a quadruple $\langle W, D, \leq, V \rangle$, where W is a non-empty set (of states); $D \subseteq W$ (the distinguished states); \leq is a partial order on W such that for all $\delta \in D$ and all $w \in W \setminus D$, $\delta \leq w$. Moreover, $V(w, p) \mapsto \{0, 1\}$ an assignment of truth values to state-variable pairs with the condition that $V(w_1, p) = 1$ and $w_1 \leq w_2$ only if $V(w_2, p) = 1$ for all $p \in \text{Prop}$ and all $w_1, w_2 \in W$. Valuations V are then extended to interpretations I to state-formula pairs as in Definition 2.2 except for the empirical negation, we extend it by the following conditions:

$$I(w, \sim A) = 1 \text{ iff for some } \delta \in D, I(\delta, A) = 0$$

Finally, semantic consequence is now defined in terms of truth preservation at all the elements of D .

¹⁰We would like to thank Thomas Müller for suggesting that models have a set of base states.

Definition 4.2 We say that A is true in a model $\mathcal{M} = \langle W, D, \leq, V \rangle$ iff for all $\delta \in D$, $V(\delta, A) = 1$. Consequence is then defined as truth preservation in a model: $\Gamma \models A$ iff for all models \mathcal{M} and all $B \in \Gamma$, if B is true in \mathcal{M} then A is true in \mathcal{M} .

It turns out, interestingly, that the “many base states” interpretation of empirical negation is equivalent to the original interpretation in the sense that they generate the same consequence relation.

Theorem 4.3 For $\Gamma \cup \{A\} \subseteq \text{Form}$, $\Gamma \models' A$ iff $\Gamma \vdash A$.

Proof. The only non-trivial direction concerns soundness, as the completeness proof is exactly the same. (We only ever need a counter model in which D is a singleton.) Showing soundness is not hard, which we leave to the interested reader. \square

Remark 4.4 If we change the truth conditions of \sim in Definition 4.1 by replacing the existential quantifier by a universal one, we obtain a strictly weaker logic. For every original model is a “universal model”, but universal models give us additional countermodels. For instance, $A \vee \sim A$ is no longer valid over the class of universal models, though it was valid on the original semantics.

4.2 Williamson on empirical negation

Williamson argues in [21] that using empirical negation \sim to block the argument to the **IPL**-inconsistency of ‘ A will never be decided’ won’t work because $\sim A \rightarrow \neg A$ ought to be valid. But then the statement that A will never be decided, $\sim(KA \vee K\neg A)$, implies $\neg(KA \vee K\neg A)$ and we’re back in inconsistency.¹¹

Why think empirical negation is stronger than intuitionistic negation, i.e. that $\sim A \rightarrow \neg A$? Williamson says:

For if \sim is to count intuitionistically as any sort of negation at all, $\sim A$ should at least be inconsistent with A in the ordinary intuitionistic sense. A warrant for $A \wedge \sim A$ should be impossible. That is, we should have $\neg(A \wedge \sim A)$. [21, p. 139]

There is good reason to believe everything he says *except for the last line*: $A \wedge \sim A$ may be impossible while $\neg(A \wedge \sim A)$ not be valid. But how?

Suppose that $\sim A$ is read ‘ A is not warranted by our current state of evidence’, as we have been interpreting it. Then $A \wedge \sim A$ can only be warranted by our current evidential state if A is currently warranted and is currently not, which is a contradiction. So $A \wedge \sim A$ could never be warranted, *except in some merely possible, non-current evidential state*. We will never be in such a state, since we ever only have our current evidence to work with, but there’s no reason to dismiss such states as playing a role in the truth conditions of empirical discourse; for they represent ways in which the current evidential state might evolve. There is a good sense, then, in which $A \wedge \sim A$ is impossible—it could

¹¹Williamson gives a standard Fitch argument for the validity of $\neg KA \rightarrow \neg A$. With this $\neg KA \wedge \neg K\neg A$, which is equivalent to $\neg(KA \vee K\neg A)$, implies $\neg A \wedge \neg\neg A$.

never be warranted by our current state of evidence—though this does not imply that $\neg(A \wedge \sim A)$ which says that $A \wedge \sim A$ could never be warranted even by *future* evidential states. But surely it could! For in some future state, we may have that A is warranted together with it being warranted that A is not warranted at our current state. In sum, we may have both $A, \sim A \vdash \perp$ and $\not\vdash \neg(A \wedge \sim A)$.¹²

Empirical negation therefore provides us with a well-motivated way of blocking a Fitch-like argument against the sort of verificationism we have been considering.

5 Conclusion

Here is a brief summary of the paper. We first discussed the motivation for empirical negation (§1) in a broadly verificationist setting. We then introduced the semantics and the proof theory (§2) which was followed by the proof of strong completeness using techniques from relevant logics (§3). The completeness proof was followed by some reflections on the semantics we presented, and on some arguments of Williamson against the kind of empirical negation we have been considering (§4). Before closing, we wish to briefly sketch two future directions of this research.

Adding empirical negation to Nelson’s logic. We have expanded intuitionistic logic by empirical negation, but there is another well-known expansion of intuitionistic logic by strong negation which results in Nelson’s logic **N3**. (cf. [20,13]) The motivation here is to consider not only the verification but also the falsification of a sentence. While empirical negation allows us to extend our constructive interpretation from the mathematical to the empirical domain, the focus is on verification. It is of interest, however, to see how we might further extend that interpretation by treating verification and falsification separately.

In formulating the semantics of Nelson’s logic, we need cover not only the truth but also the falsity conditions of sentences. Since we already have the truth condition for empirical negation, what we need in carrying out our extension are adequate falsity conditions for empirical negation. We shall, however, have to leave these details for another occasion.

Empirical negation as classical negation. One way of looking at empirical negation is as classical negation; in other words, \mathbf{IPC}^\sim can be seen as an expansion of intuitionistic logic by classical negation. Some expansions in a similar vein may be found in [5,11,12].¹³ To see the difference between our account and others, consider Kripke semantics for intuitionistic logic along with the most straightforward truth condition for classical negation:

$$I(w, \sim A) = 1 \text{ iff } I(w, A) = 0. \quad (\star)$$

Validity is defined as truth preservation at *all* states, not just some distinguished state(s). A natural question then is to ask whether extending intu-

¹²This is part of the result proved in Proposition 2.15.

¹³See also [1] in which intuitionistic and classical implications are combined.

intuitionistic logic this way, in terms of (\star) , is better or worse compared with extending it by empirical negation. While this is a question we wish to address in a more general setting on another occasion, let us quickly point out a fact that favors the empirical negation approach in an intuitionistic context.

If we define classical negation by (\star) , the resulting system conservatively extends classical logic but it is not sound for intuitionistic logic (since truth is no longer preserved up the order). On the other hand, if we define classical negation as empirical negation, then the resulting system conservatively extends the intuitionistic fragment while preserving soundness. Empirical negation may then be seen as providing a simple way of adding classical negation to intuitionistic logic or any logic whose semantics makes use of a similar kind of requirement that truth be preserved up an order, such as a simplified semantics for relevant logic. Indeed, we may regard the semantics for empirical negation as a special case of so-called star semantics for relevant logics given by Richard and Val Routley in [17].¹⁴ In this case, the star function maps every world to a single world, namely the base world g . We leave the task of investigating this connection to relevant logic for another occasion.

Appendix

Proof of Lemma 3.8:

Let A_0, A_1, \dots be an enumeration of the set of formulas Form . Then we define Σ_i and Δ_i ($i \in \omega$) by induction as follows:

- $\Sigma_0 := \Sigma$; $\Delta_0 := \Delta$.
- For Σ_i , the definition is as follows:

$$\Sigma_{i+1} := \begin{cases} \Sigma_i \cup \{A_i\} & \text{if } \not\vdash_{\pi} \Sigma_i \cup \{A_i\} \rightarrow \Delta_i \\ \Sigma_i & \text{otherwise} \end{cases}$$

And for Δ_i , the definition is as follows:

$$\Delta_{i+1} := \begin{cases} \Delta_i & \text{if } \not\vdash_{\pi} \Sigma_i \cup \{A_i\} \rightarrow \Delta_i \\ \Delta_i \cup \{A_i\} & \text{otherwise} \end{cases}$$

Finally, we define Σ', Δ' as follows:

$$\Sigma' := \bigcup_{i < \omega} \Sigma_i \text{ and } \Delta' := \bigcup_{i < \omega} \Delta_i.$$

We now prove that $\langle \Sigma', \Delta' \rangle$ is a Π -partition. For this purpose, we need to prove the following two facts:

- (i) $\Sigma' \cup \Delta' = \text{Form}$
- (ii) $\not\vdash_{\pi} \Sigma' \rightarrow \Delta'$

¹⁴We thank a referee for this interesting remark.

Since the former is satisfied by the construction, it need only show the latter. By the compactness of \vdash_π , if $\vdash_\pi \Sigma' \rightarrow \Delta'$ then $\vdash_\pi \Sigma_k \rightarrow \Delta_k$ for some $k \in \omega$. Therefore, it suffices to show that for no $i \in \omega$, $\not\vdash_\pi \Sigma_i \rightarrow \Delta_i$. This can be proved by induction on i . For the case when $i = 0$, it is true by the definition. Suppose now that it is true for $i = j$ but not $i = j + 1$. Then, we have $\not\vdash_\pi \Sigma_j \rightarrow \Delta_j$ and $\vdash_\pi \Sigma_{j+1} \rightarrow \Delta_{j+1}$.

Now, if $\vdash_\pi \Sigma_{j+1} \rightarrow \Delta_{j+1}$ and $\not\vdash_\pi \Sigma_j \cup \{A_j\} \rightarrow \Delta_j$, then $\Sigma_{j+1} = \Sigma_j \cup \{A_j\}$ and $\Delta_{j+1} = \Delta_j$ by the latter, and thus by applying this to the former we obtain $\vdash_\pi \Sigma_j \cup \{A_j\} \rightarrow \Delta_j$. Therefore, $\vdash_\pi \Sigma_{j+1} \rightarrow \Delta_{j+1}$ implies $\vdash_\pi \Sigma_j \cup \{A_j\} \rightarrow \Delta_j$ by reductio. So, we obtain the following from $\vdash_\pi \Sigma_{j+1} \rightarrow \Delta_{j+1}$:

$$\vdash_\pi \Sigma_j \cup \{A_j\} \rightarrow \Delta_j$$

On the other hand, if $\vdash_\pi \Sigma_{j+1} \rightarrow \Delta_{j+1}$ and $\vdash_\pi \Sigma_j \cup \{A_j\} \rightarrow \Delta_j$, then $\Sigma_{j+1} = \Sigma_j$ and $\Delta_{j+1} = \Delta_j \cup \{A_j\}$ by the latter, and thus by applying this to the former we obtain $\vdash_\pi \Sigma_j \rightarrow \Delta_j \cup \{A_j\}$. Therefore, by the above result and $\vdash_\pi \Sigma_{j+1} \rightarrow \Delta_{j+1}$, we get the following:

$$\vdash_\pi \Sigma_j \rightarrow \Delta_j \cup \{A_j\}$$

So, for some conjunctions C_1, C_2 of members of Σ_j and some disjunctions D_1, D_2 of members of Δ_j , we obtain the following:

$$C_1 \wedge A_j \rightarrow D_1 \quad C_2 \rightarrow D_2 \vee A_j.$$

But then, this leads to contradiction. Indeed,

1	$C_1 \wedge A_j \rightarrow D_1$	[sup.]
2	$C_2 \rightarrow D_2 \vee A_j$	[sup.]
3	$C_1 \wedge C_2 \rightarrow C_2$	[(Ax4)]
4	$C_1 \wedge C_2 \rightarrow D_2 \vee A_j$	[3, 2, (Transitivity)]
5	$C_1 \wedge C_2 \rightarrow C_1$	[(Ax3)]
6	$C_1 \wedge C_2 \rightarrow (D_2 \vee A_j) \wedge C_1$	[4, 5, (Ax5), (MP)]
7	$C_1 \wedge C_2 \rightarrow D_2 \vee (A_j \wedge C_1)$	[6, (6), (MP)]
8	$C_1 \wedge A_j \rightarrow D_1 \vee D_2$	[1, (Ax6), (MP)]
9	$D_2 \rightarrow D_1 \vee D_2$	[(Ax7)]
10	$D_2 \vee (C_1 \wedge A_j) \rightarrow D_1 \vee D_2$	[8, 9, (Ax8), (MP)]
11	$C_1 \wedge C_2 \rightarrow D_1 \vee D_2$	[7, 10, (Transitivity)]

And this last formula shows that $\vdash_\pi \Sigma_j \rightarrow \Delta_j$ which is a contradiction in view of $\not\vdash_\pi \Sigma_j \rightarrow \Delta_j$.

Proof of Lemma 3.10:

Let A_0, A_1, \dots be an enumeration of the set of formulas **Form**. Then we define Σ_i and Δ_i ($i \in \omega$) by induction as follows:

- $\Sigma_0 := \Sigma$; $\Delta_0 := \Delta$.
- For Σ_i , the definition is as follows:

$$\Sigma_{i+1} := \begin{cases} \Sigma_i \cup \{A_i\} & \text{if } \Sigma_i \cup \{A_i\} \not\vdash \Delta_i \\ \Sigma_i & \text{otherwise} \end{cases}$$

And for Δ_i , the definition is as follows:

$$\Delta_{i+1} := \begin{cases} \Delta_i & \text{if } \Sigma_i \cup \{A_i\} \not\vdash \Delta_i \\ \Delta_i \cup \{A_i\} & \text{otherwise} \end{cases}$$

Finally, we define Σ', Δ' as follows:

$$\Sigma' := \bigcup_{i < \omega} \Sigma_i \text{ and } \Delta' := \bigcup_{i < \omega} \Delta_i.$$

Then, we now prove that $\langle \Sigma', \Delta' \rangle$ is a *partition*. For this purpose, we need to prove the following two facts:

- (i) $\Sigma' \cup \Delta' = \text{Form}$
- (ii) $\not\vdash \Sigma' \rightarrow \Delta'$

Since the former is satisfied by the construction, only the latter remains to be shown. By the compactness of \vdash , if $\vdash \Sigma' \rightarrow \Delta'$ then $\vdash \Sigma_k \rightarrow \Delta_k$ for some $k \in \omega$. Furthermore, it holds that $\vdash \Sigma \rightarrow \Delta$ implies $\Sigma \vdash \Delta$ for any Σ and Δ . Therefore, it suffices to show that for no $i \in \omega$, $\Sigma_i \vdash \Delta_i$. This can be proved by induction on i . For the case when $i = 0$, it is true by the definition. Suppose now that it is true for $i = j$ but not $i = j + 1$. Then, we have $\Sigma_j \not\vdash \Delta_j$ and $\Sigma_{j+1} \vdash \Delta_{j+1}$.

Case 1. if $\Sigma_j \cup \{A_j\} \not\vdash \Delta_j$, then $\Sigma_{j+1} = \Sigma_j \cup \{A_j\}$ and $\Delta_{j+1} = \Delta_j$, and thus by applying this to $\Sigma_{j+1} \vdash \Delta_{j+1}$ we obtain $\Sigma_j \cup \{A_j\} \vdash \Delta_j$. Therefore, $\Sigma_j \cup \{A_j\} \vdash \Delta_j$ by reductio.

Case 2. if $\Sigma_j \cup \{A_j\} \vdash \Delta_j$, then $\Sigma_{j+1} = \Sigma_j$ and $\Delta_{j+1} = \Delta_j \cup \{A_j\}$ by the latter, and thus by applying this to $\Sigma_{j+1} \vdash \Delta_{j+1}$ we obtain $\Sigma_j \vdash \Delta_j \cup \{A_j\}$. Therefore, by the above result, we get $\Sigma_j \vdash \Delta_j \cup \{A_j\}$.

So, for some conjunctions C_1, C_2 of members of Σ_j and some disjunctions D_1, D_2 of members of Δ_j , we obtain the following:

$$C_1 \wedge A_j \vdash D_1 \quad C_2 \vdash D_2 \vee A_j.$$

But then, this leads to contradiction. Indeed,

- | | | |
|----|---|-------------------------------|
| 1 | $C_1 \wedge A_j \vdash D_1$ | [sup.] |
| 2 | $C_2 \vdash D_2 \vee A_j$ | [sup.] |
| 3 | $C_1 \wedge C_2 \vdash C_2$ | [(Ax4)] |
| 4 | $C_1 \wedge C_2 \vdash D_2 \vee A_j$ | [3, 2, (Transitivity)] |
| 5 | $C_1 \wedge C_2 \vdash C_1$ | [(Ax3)] |
| 6 | $C_1 \wedge C_2 \vdash (D_2 \vee A_j) \wedge C_1$ | [4, 5, (Ax5), (MP)] |
| 7 | $C_1 \wedge C_2 \vdash D_2 \vee (A_j \wedge C_1)$ | [6, (6), (MP)] |
| 8 | $C_1 \wedge A_j \vdash D_1 \vee D_2$ | [1, (Ax6), (MP)] |
| 9 | $D_2 \vdash D_1 \vee D_2$ | [(Ax7)] |
| 10 | $D_2 \vee (C_1 \wedge A_j) \vdash D_1 \vee D_2$ | [8, 9, Proposition 3.4, (MP)] |
| 11 | $C_1 \wedge C_2 \vdash D_1 \vee D_2$ | [7, 10, (Transitivity)] |

And this last formula shows that $\Sigma_j \vdash \Delta_j$ which is a contradiction in view of $\Sigma_j \not\vdash \Delta_j$.

All that remains is to show that Σ' is deductively closed. Suppose that $\Sigma' \vdash A$ but $A \notin \Sigma'$. Since $\Sigma' \cup \Delta' = \text{Form}$, we have $A \in \Delta'$, and thus $\Sigma' \vdash \Delta'$. But this is a contradiction. \square

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