Before announcement

Philippe Balbiani [1]

Institut de recherche en informatique de Toulouse
Toulouse University

Hans van Ditmarsch [2]

Laboratoire lorrain de recherche en informatique et ses applications
CNRS — Université de Lorraine — Inria

Andreas Herzig[1]

Institut de recherche en informatique de Toulouse
Toulouse University

Abstract

We axiomatize the mono-agent logic of knowledge with public announcements and converse public announcements. A special variant of our logic is determined by the model of maximal ignorance wherein the agent considers all valuations of atomic formulas possible.

Keywords: Public announcement logic, subset space logic, complete axiomatization.

1 Introduction

Public Announcement Logic [15] models the effect of publicly observable events on information states, $[\phi]^+\psi$ standing for "after ϕ's announcement, ψ is true". Now, suppose you want to go in the other direction, $[\phi]^-\psi$ standing for "before ϕ's announcement, ψ was true". Just as $[\phi]^+$ has a diamond-version noted $\langle\phi\rangle^+$, $[\phi]^-$ also has a diamond-version which we note $\langle\phi\rangle^-$. Surely the relation between public announcement and converse public announcement resembles the behaviour of future and past constructs in temporal logic. Indeed, one expects the validity of $\psi \to [\phi]^+\langle\phi\rangle^-\psi$ and $\psi \to [\phi]^-\langle\phi\rangle^+\psi$. However, unlike the announcement operation, the converse announcement operation is not deterministic: different states of information may lead to the same outcome state

[1] IRIT, Toulouse University, 118 route de Narbonne, 31062 Toulouse Cedex 9, FRANCE; {Philippe.Balbiani,Andreas.Herzig}@irit.fr.

[2] LORIA, CNRS — Université de Lorraine — Inria, Campus scientifique, BP 239, 54506 Vandœuvre-lès-Nancy Cedex, FRANCE; hans.van-ditmarsch@loria.fr. Hans van Ditmarsch is also affiliated to IMSc, Chennai, India and to Zhejiang University, China.

of information.
Dynamic epistemic logics with constructs to denote what was the case before the executions of actions have been investigated in [1,18] where they are interpreted over history-based structures. See also [11,20]. In such structures, what agents currently know are mere snapshots forming parts of a larger structure that also contains possible prior and posterior states of information. These states can then be accessed in some way. In [18], this access is realized with the history-based structures of [14]. An actual state is accompanied by a list of prior actions: going back in the past means removing the last event from that history. This changed perspective then makes it possible to check what was true before now. In [16,17], this history-based approach has also been generalized to cover non-public events.
In dynamic epistemic logics, fresh atomic formulas can be used to store the values of formulas. This is useful, as the denotation of atomic formulas is constant throughout action execution but the denotation of formulas is not. This feature is used in the satisfiability preserving transformations demonstrating complexity arguments in dynamic epistemic logic [12]. It is used as well for the purpose of keeping denotations constant, for instance in [8] where the authors model the announcement "I knew that you know the number pair" in the Sum-and-Product riddle. Similarly, this technique is used to model "Do not turn on the light if you have turned it on already in the past" in the "One hundred prisoners and a light bulb" riddle [6].
As discussed above, the converse announcement operation is not deterministic and has similarities with quantification: in a given model, a converse announcement can be interpreted as one of the multifarious expansions of that model satisfying ϕ. About quantification, two analogues come to mind: Arbitrary Public Announcement Logic [2] and Refinement Modal Logic [4]. In Arbitrary Public Announcement Logic, $\blacksquare^+\phi$ stands for "ϕ is true after any arbitrary announcement". This quantifies over all modally definable restrictions of the actual information state whereas in converse announcements we quantify over all expansions containing the actual information state. In Refinement Modal Logic, $\blacksquare^+\phi$ means that ϕ is true in any structure that is a modal refinement of the actual information state.
A less likely place to look for reasoning about the past is in the setting of Subset Space Logic [5,13]. Apart from the epistemic modalities $\Box$ and $\Diamond$, which behave like $S5$-modalities, we now also have the so-called effort modalities $\blacksquare^+$ and $\blacklozenge^+$, which behave like $S4$-modalities. A typical schema in Subset Space Logic is $\blacklozenge^+\Box\phi$ which stands for "after some effort, the agent knows that ϕ". This logic is interpreted on models consisting of a domain plus a set of "enabled" subsets of the domain. A formula is true after some effort if there is an enabled subset of the current set that satisfies it. In Subset Space Logics, converse $\blacksquare^-$ and $\blacklozenge^-$ effort modalities were also investigated [9]. In Subset Space Logics with converse, we can now formalize statements like "before some effort, the agent knew that ϕ" by $\blacklozenge^-\Box\phi$.
An appealing semantics for converse announcement of ϕ is that of truth in all

models of which the current model is the ϕ restriction. We did not manage to axiomatize that logic. Instead, we chose a setting similar to that of Subset Space Logic with converse: given a state-subset pair in a model, a formula ψ is true before announcement of ϕ if ψ is true for all state-subset pairs in that model whose subset component contains the corresponding component in the given pair. This semantics comes at the price of losing some validities of the previous semantics, such as $\Box p \rightarrow \langle p \rangle^- \Diamond \neg p$. However, we can recover some of our desiderata in the largest subset space model: the model consisting of all valuations and all subsets of that. One possible logic of "before announcement" is then the special case of the theory of that model.
The section-by-section breakdown of the paper is as follows. In Sections 2 and 3, we present the syntax and the semantics of a mono-agent logic of knowledge with public announcements and converse public announcements. The aim of Section 4 is to demonstrate that the constructs $[\cdot]^+$ and $[\cdot]^-$ cannot be eliminated from our language. In Section 5, we compare our mono-agent logic to subset space logic. In Section 6, we give an axiomatization and in Sections 7 and 8, we prove its completeness. The purpose of Section 9 is to analyse public announcements and converse public announcements in the largest subset space model. Easy proofs have been omitted whereas some others can be found in the Annex.

2 Syntax

Let VAR be a countable set of atomic formulas (with typical members p, q, etc). The formulas are inductively defined as follows:

- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi \mid [\phi]^+\psi \mid [\phi]^-\psi$.

We define the other Boolean constructs as usual. The formulas $\Diamond\phi$, $\langle\phi\rangle^+\psi$ and $\langle\phi\rangle^-\psi$ are obtained as the following abbreviations: $\Diamond\phi$ is $\neg\Box\neg\phi$, $\langle\phi\rangle^+\psi$ is $\neg[\phi]^+\neg\psi$ and $\langle\phi\rangle^-\psi$ is $\neg[\phi]^-\neg\psi$. For the collection of boxes, we propose the following readings:

- $\Box\phi$: "the agent considers it necessary according to her knowledge that ϕ",
- $[\phi]^+\psi$: "every execution of the announcement ϕ that comes from the present situation leads to a situation bearing ψ",
- $[\phi]^-\psi$: "every execution of the announcement ϕ that leads to the present situation comes from a situation bearing ψ".

For the collection of diamonds, we propose the following readings:

- $\Diamond\phi$: "the agent considers it possible according to her knowledge that ϕ",
- $\langle\phi\rangle^+\psi$: "some execution of the announcement ϕ coming from the present situation leads to a situation bearing ψ",
- $\langle\phi\rangle^-\psi$: "some execution of the announcement ϕ leading to the present situation comes from a situation bearing ψ".

The key point to note about the constructs $[\cdot]^+$ and $[\cdot]^-$ is that they allow to make modalities out of formulas. Our language can be used to reason about the knowledge of some agent after and before announcements are executed. Let ϕ^0 and ϕ^1 respectively denote the formulas $\neg\phi$ and ϕ. We adopt the usual rules for omission of the parentheses. Let $(p_1, p_2, \ldots)$ be a non-repeating enumeration of VAR. Let $k \in \mathbb{N}$. A k-formula is a formula whose atomic formulas form a sublist of $(p_1, \ldots, p_k)$. A k-world is a formula of the form $p_1^{a_1} \wedge \ldots \wedge p_k^{a_k}$ where $a_1, \ldots, a_k \in \{0, 1\}$ and a k-step is a non-empty set of k-worlds. Obviously, there exist exactly 2^k k-worlds and there exist exactly $2^{2^k} - 1$ k-steps. A pair (α, A) consisting of a k-world α and a k-step A such that $\alpha \in A$ is called a k-tip. For all sets Γ of formulas, let

- $\Box\Gamma = \{\phi : \Box\phi \in \Gamma\}$,
- $[\phi]^+\Gamma = \{\psi : [\phi]^+\psi \in \Gamma\}$,
- $[\phi]^-\Gamma = \{\psi : [\phi]^-\psi \in \Gamma\}$.

The degree of a formula ϕ, in symbols $deg(\phi)$, is inductively defined as follows:

- $deg(p) = 3$,
- $deg(\bot) = 3$,
- $deg(\neg\phi) = deg(\phi)$,
- $deg(\phi \vee \psi) = \max\{deg(\phi), deg(\psi)\}$,
- $deg(\Box\phi) = deg(\phi)$,
- $deg([\phi]^+\psi) = deg(\phi) + deg(\psi)$,
- $deg([\phi]^-\psi) = deg(\phi) + deg(\psi)$.

Let the size of a formula ϕ, in symbols $size(\phi)$, be the number of occurrences of symbols it contains. For all finite sets Γ of formulas and for all formulas ψ, the formulas $\bigvee\Gamma$, $\nabla\Gamma$ and $\nabla_\psi\Gamma$ are defined by the following abbreviations:

- $\bigvee\Gamma ::= \bigvee\{\phi : \phi \in \Gamma\}$,
- $\nabla\Gamma ::= \Box\bigvee\Gamma \wedge \bigwedge\{\Diamond\phi : \phi \in \Gamma\}$,
- $\nabla_\psi\Gamma ::= \psi \wedge \Box(\psi \to \bigvee\Gamma) \wedge \bigwedge\{\Diamond(\psi \wedge \phi) : \phi \in \Gamma\}$.

3 Semantics

A model is a triple of the form $\mathcal{M} = (W, X, V)$ where W is a non-empty set (with typical members x, y, etc), X is a non-empty set of non-empty subsets of W (with typical members S, T, etc) and V is a function associating to each $p \in VAR$ a subset $V(p)$ of W. Elements of W will be called worlds. Each of them is an epistemic alternative to the real world: due to her lack of knowledge, the agent is not able to distinguish between the real world and its epistemic alternatives. Elements of X will be called steps. Each of them contains the real world together with its epistemic alternatives at some moment of their history. We shall say that a world-step pair (x, S) is a tip iff $x \in S$. Each tip determines the real world and the current restriction of the model containing the real world together with its epistemic alternatives. To flesh this out a little, the universal relation in the step component of tips should be interpreted as a contemporaneity relation between moments. As for the function V, it assigns to all atomic formulas, the set of all $\mathcal{M}$-worlds in which it holds. It will be called valuation of $\mathcal{M}$. The satisfiability of a formula ϕ in a model $\mathcal{M} = (W, X, V)$ at tip (x, S), in symbols $\mathcal{M}, (x, S) \models \phi$, is inductively defined as follows:

- $\mathcal{M}, (x, S) \models p$ iff $x \in V(p)$,
- $\mathcal{M}, (x, S) \not\models \bot$,
- $\mathcal{M}, (x, S) \models \neg\phi$ iff $\mathcal{M}, (x, S) \not\models \phi$,
- $\mathcal{M}, (x, S) \models \phi \vee \psi$ iff $\mathcal{M}, (x, S) \models \phi$ or $\mathcal{M}, (x, S) \models \psi$,
- $\mathcal{M}, (x, S) \models \Box\phi$ iff for all $y \in S$, $\mathcal{M}, (y, S) \models \phi$,
- $\mathcal{M}, (x, S) \models [\phi]^{+}\psi$ iff for all $T \in X$, if $x \in T$ and $T = \{z \in S : \mathcal{M}, (z, S) \models \phi\}$ then $\mathcal{M}, (x, T) \models \psi$,
- $\mathcal{M}, (x, S) \models [\phi]^{-}\psi$ iff for all $T \in X$, if $x \in T$ and $S = \{z \in T : \mathcal{M}, (z, T) \models \phi\}$ then $\mathcal{M}, (x, T) \models \psi$.

Let us reflect upon these truth conditions. First, the satisfiability of an atomic formula does not depend on the step components of tips: it only depends on their world components. This indicates that the concept of validity that we will define at the end of this section will give rise to a non-normal set of valid formulas. Second, the Boolean constructs are classically interpreted. Third, the $\Box$ construct behaves like a universal modality in the step component of tips. This makes it similar to the epistemic construct in subset space logic. Fourth, the $[\cdot]^{+}$ construct behaves like an announcement modality: if a tip satisfies ϕ then $[\phi]^{+}$ further restricts the model to those epistemic alternatives in the step component of the tip satisfying ϕ. As a result, the announcement modality $[\cdot]^{+}$ is always deterministic. The reader may have understood from the definition that a true announcement is executable in a tip iff the above-mentioned restriction of the model is itself a step. With the concept of freedom that we will define at the end of this section, we will concentrate on the class of all models in which true announcements are always executable. Fifth, the $[\cdot]^{-}$ construct behaves like the converse of the announcement modality $[\cdot]^{+}$. Seeing that the above-mentioned restriction of the model to those epistemic alternatives in the step components of differents tips satisfying an announced formula may produce the same result, there is no reason to expect the $[\cdot]^{-}$ construct to be deterministic. In any case, obviously,

- $\mathcal{M}, (x, S) \models \Diamond\phi$ iff there exists $y \in S$ such that $\mathcal{M}, (y, S) \models \phi$,
- $\mathcal{M}, (x, S) \models \langle\phi\rangle^{+}\psi$ iff there exists $T \in X$ such that $x \in T$, $T = \{z \in S : \mathcal{M}, (z, S) \models \phi\}$ and $\mathcal{M}, (x, T) \models \psi$,
- $\mathcal{M}, (x, S) \models \langle\phi\rangle^{-}\psi$ iff there exists $T \in X$ such that $x \in T$, $S = \{z \in T : \mathcal{M}, (z, T) \models \phi\}$ and $\mathcal{M}, (x, T) \models \psi$.

Let $R_{\Box}^{\mathcal{M}}$ be the binary relation between tips such that $(x, S)\ R_{\Box}^{\mathcal{M}}\ (y, T)$ iff $S = T$. Obviously, $R_{\Box}^{\mathcal{M}}$ is an equivalence relation between tips such that: $\mathcal{M}, (x, S) \models \Box\phi$ iff for all tips (y, T), if $(x, S)\ R_{\Box}^{\mathcal{M}}\ (y, T)$ then $\mathcal{M}, (y, T) \models \phi$. Let $R_{[\phi]^{+}}^{\mathcal{M}}$ and $R_{[\phi]^{-}}^{\mathcal{M}}$ be the binary relations between tips such that (i) $(x, S)\ R_{[\phi]^{+}}^{\mathcal{M}}\ (y, T)$ iff $x = y$ and $T = \{z \in S : \mathcal{M}, (z, S) \models \phi\}$ and

(ii) $(x, S)\ R^{\mathcal{M}}_{[\phi]^-}\ (y, T)$ iff $x = y$ and $S = \{z \in T : \mathcal{M}, (z, T) \models \phi\}$. Obviously, $R^{\mathcal{M}}_{[\phi]^+}$ and $R^{\mathcal{M}}_{[\phi]^-}$ are mutually converse relations between tips such that: (i) $\mathcal{M}, (x, S) \models [\phi]^+\psi$ iff for all tips (y, T), if $(x, S)\ R^{\mathcal{M}}_{[\phi]^+}\ (y, T)$ then $\mathcal{M}, (y, T) \models \phi$, (ii) $\mathcal{M}, (x, S) \models [\phi]^-\psi$ iff for all tips (y, T), if $(x, S)\ R^{\mathcal{M}}_{[\phi]^-}\ (y, T)$ then $\mathcal{M}, (y, T) \models \phi$. Moreover, the binary relation $R^{\mathcal{M}}_{[\phi]^+}$ is always deterministic. Remark that $R^{\mathcal{M}}_{[\top]^+}$ and $R^{\mathcal{M}}_{[\top]^-}$ are equal to the identity relation between tips. Let $\equiv^{\mathcal{M}}$ be the transitive closure of $\bigcup\{R^{\mathcal{M}}_{[\phi]^+} : \phi$ is a formula$\} \cup \bigcup\{R^{\mathcal{M}}_{[\phi]^-} : \phi$ is a formula$\}$. Obviously, $\equiv^{\mathcal{M}}$ is an equivalence relation between tips. Let $\equiv^{\mathcal{M}}_{\Box}$ be the transitive closure of $R^{\mathcal{M}}_{\Box} \cup \equiv^{\mathcal{M}}$. Obviously, $\equiv^{\mathcal{M}}_{\Box}$ is an equivalence relation between tips. Moreover, $\equiv^{\mathcal{M}}_{\Box}$ is coarser than $R^{\mathcal{M}}_{\Box}$ and $\equiv^{\mathcal{M}}$. We shall say that a formula ϕ is globally true in a model $\mathcal{M}$, in symbols $\mathcal{M} \models \phi$, if ϕ is satisfied at all tips in $\mathcal{M}$. There are two ways for the announcement of ϕ to fail in a model $\mathcal{M} = (W, X, V)$ at tip (x, S). One is for ϕ to be false at (x, S) (which matches the traditional semantics of announcements). The other is for the set of worlds in S for which ϕ is true not to be in X. A model $\mathcal{M} = (W, X, V)$ is said to be free if for all formulas ϕ and for all tips (x, S), if $\mathcal{M}, (x, S) \models \phi$ then there exists $T \in X$ such that $x \in T$ and $T = \{z \in S : \mathcal{M}, (z, S) \models \phi\}$. Obviously, in free models, true announcements are executable. Moreover, any model $\mathcal{M} = (W, X, V)$ in which X is closed under non-empty subsets is free. However, note that, in a free model $\mathcal{M} = (W, X, V)$, X is not necessarily closed under subsets.

Lemma 3.1 *Let $\mathcal{M}$ be a model. $\mathcal{M}$ is free iff for all formulas ϕ, $\mathcal{M} \models \phi \rightarrow \langle\phi\rangle^+\top$.*

A formula ϕ is said to be valid, in symbols $\models \phi$, if ϕ is globally true in all free models. In Sections 6–8, we will give a complete axiomatization of the set of all valid formulas. In the meantime, it is well worth noting some interesting properties.

Lemma 3.2 *The following formulas are valid:*

- $[\phi]^+p \leftrightarrow (\phi \rightarrow p)$,
- $[\phi]^+\bot \leftrightarrow \neg\phi$,
- $[\phi]^+\neg\psi \leftrightarrow (\phi \rightarrow \neg[\phi]^+\psi)$,
- $[\phi]^+(\psi \vee \chi) \leftrightarrow [\phi]^+\psi \vee [\phi]^+\chi$,
- $[\phi]^+\Box\psi \leftrightarrow (\phi \rightarrow \Box[\phi]^+\psi)$.

Lemma 3.3 *Let ϕ be a formula. If ϕ is $\{[\cdot]^+, [\cdot]^-\}$-free then $\models \phi$ iff $\phi \in S5$.*

Lemma 3.4 *Let ϕ be a formula. If ϕ is $[\cdot]^-$-free then $\models \phi$ iff $\phi \in PAL$.*

4 Expressivity

We tackle the problem of the definability of $[\cdot]^+$ and $[\cdot]^-$ in the class of all free models.

Proposition 4.1 (i) $[\cdot]^+$ *cannot be eliminated from the language in the class of all free models.*

(ii) $[\cdot]^-$ *cannot be eliminated from the language in the class of all free models.*

Proof. (1) Suppose $[\cdot]^+$ can be eliminated from the language in the class of all free models. Hence, there exists a formula $\phi(p,q)$ in $\Box$ and $[\cdot]^-$ such that $(*)$ for all free models $\mathcal{M} = (W, X, V)$ and for all tips (x, S), $\mathcal{M}, (x, S) \models \langle p\rangle^+\langle q\rangle^-\Diamond(p \wedge \neg q)$ iff $\mathcal{M}, (x, S) \models \phi(p, q)$. Let $\mathcal{M} = (W, X, V)$ and $\mathcal{M}' = (W', X', V')$ be the models such that $W = W' = \{x, y, z\}$, $X = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}\}$, $X' = \{\{x\}, \{y\}, \{z\}, \{x, y\}\}$, $V(p) = V'(p) = \{y, z\}$ and $V(q) = V'(q) = \{y\}$. Obviously, $\mathcal{M}$ and $\mathcal{M}'$ are free. Moreover, $\mathcal{M}, (y, \{x, y\}) \models \langle p\rangle^+\langle q\rangle^-\Diamond(p \wedge \neg q)$ and $\mathcal{M}', (y, \{x, y\}) \not\models \langle p\rangle^+\langle q\rangle^-\Diamond(p \wedge \neg q)$. By $(*)$, $\mathcal{M}, (y, \{x, y\}) \models \phi(p, q)$ and $\mathcal{M}', (y, \{x, y\}) \not\models \phi(p, q)$. Nevertheless, a proof by induction, based on the function $size(\cdot)$ defined in Section 2, would lead to the conclusion that for all formulas $\psi(p, q)$ in $\Box$ and $[\cdot]^-$, $\mathcal{M}, (y, \{x, y\}) \models \psi(p, q)$ iff $\mathcal{M}', (y, \{x, y\}) \models \psi(p, q)$.
(2) Suppose $[\cdot]^-$ can be eliminated from the language in the class of all free models. Hence, there exists a formula $\phi(p, q)$ in $\Box$ and $[\cdot]^+$ such that $(*)$ for all free models $\mathcal{M} = (W, X, V)$ and for all tips (x, S), $\mathcal{M}, (x, S) \models \langle p\rangle^-\Diamond q$ iff $\mathcal{M}, (x, S) \models \phi(p, q)$. Let $\mathcal{M} = (W, X, V)$ and $\mathcal{M}' = (W', X', V')$ be the models such that $W = W' = \{x, y\}$, $X = \{\{x\}, \{y\}, \{x, y\}\}$, $X' = \{\{x\}, \{y\}\}$, $V(p) = V'(p) = \{x\}$ and $V(q) = V'(q) = \{y\}$. Obviously, $\mathcal{M}$ and $\mathcal{M}'$ are free. Moreover, $\mathcal{M}, (x, \{x\}) \models \langle p\rangle^-\Diamond q$ and $\mathcal{M}', (x, \{x\}) \not\models \langle p\rangle^-\Diamond q$. By $(*)$, $\mathcal{M}, (x, \{x\}) \models \phi(p, q)$ and $\mathcal{M}', (x, \{x\}) \not\models \phi(p, q)$. Nevertheless, a proof by induction, based on the function $size(\cdot)$ defined in Section 2, would lead to the conclusion that for all formulas $\psi(p, q)$ in $\Box$ and $[\cdot]^+$, $\mathcal{M}, (x, \{x\}) \models \psi(p, q)$ iff $\mathcal{M}', (x, \{x\}) \models \psi(p, q)$. $\Box$

Proposition 4.1 implies that the constructs $[\cdot]^+$ and $[\cdot]^-$ cannot be eliminated from our language.

5 Relationships with subset space logic

Let the language be extended with the constructs $\blacksquare^+$ and $\blacksquare^-$ with diamond-versions $\blacklozenge^+$ and $\blacklozenge^-$ and let the truth-conditions of formulas $\blacksquare^+\phi$ and $\blacksquare^-\phi$ in model $\mathcal{M} = (W, X, V)$ at tip (x, S) be defined as follows:

- $\mathcal{M}, (x, S) \models \blacksquare^+\phi$ iff for all $T \in X$, if $x \in T$ and $T \subseteq S$ then $\mathcal{M}, (x, T) \models \phi$,
- $\mathcal{M}, (x, S) \models \blacksquare^-\psi$ iff for all $T \in X$, if $x \in T$ and $S \subseteq T$ then $\mathcal{M}, (x, T) \models \phi$.

Obviously, $\blacksquare^+$ is the so-called effort modality of Subset Space Logic [5,13] and $\blacksquare^-$ is the converse effort modality introduced by Heinemann [9].

Proposition 5.1 $\blacksquare^+$ *and* $\blacksquare^-$ *cannot be both eliminated from the language in the class of all models.*

Proof. Suppose $\blacksquare^+$ and $\blacksquare^-$ can be both eliminated from the language in the class of all models. Hence, there exists a formula $\phi(p, q)$ in $\Box$, $[\cdot]^+$ and $[\cdot]^-$ such that $(*)$ for all models $\mathcal{M} = (W, X, V)$ and for all tips (x, S), $\mathcal{M}, (x, S) \models \blacklozenge^-\Diamond(p \wedge \blacklozenge^+\blacklozenge^-\Diamond q)$ iff $\mathcal{M}, (x, S) \models \phi(p, q)$. Let $\mathcal{M} = (W, X, V)$ and $\mathcal{M}' = (W', X', V')$ be the models such that $W = W' = \{x, y, z, t\}$, $X =$

$\{\{x\},\{z\},\{z,t\},\{x,y,z\}\}$, $X' = \{\{x\},\{z,t\},\{x,y,z\}\}$, $V(p) = V'(p) = \{y,z\}$ and $V(q) = V'(q) = \{t\}$. Obviously, $\mathcal{M},(x,\{x\}) \models \blacklozenge^-\Diamond(p \wedge \blacklozenge^+\blacklozenge^-\Diamond q)$ and $\mathcal{M}',(x,\{x\}) \not\models \blacklozenge^-\Diamond(p \wedge \blacklozenge^+\blacklozenge^-\Diamond q)$. By $(*)$, $\mathcal{M},(x,\{x\}) \models \phi(p,q)$ and $\mathcal{M}',(x,\{x\}) \not\models \phi(p,q)$. Nevertheless, a proof by induction, based on the function $size(\cdot)$ defined in Section 2, would lead to the conclusion that for all formulas $\psi(p,q)$ in $\Box$, $[\cdot]^+$ and $[\cdot]^-$, $\mathcal{M},(x,\{x\}) \models \psi(p,q)$ iff $\mathcal{M}',(x,\{x\}) \models \psi(p,q)$.$\Box$

Proposition 5.1 implies that the constructs $\blacksquare^+$ and $\blacksquare^-$ of subset space logics cannot be both defined in our language.

6 Axiomatization

Let $PAL^{\pm}$ be the least set of formulas containing the following axioms and closed under the following inference rules:

(A_1) all instances of CPL,

(A_2) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$,

(A_3) $\Box\phi \to \phi$,

(A_4) $\Diamond\phi \to \Box\Diamond\phi$,

(A_5) $\Box\phi \to \Box\Box\phi$,

(A_6) $[\phi]^+(\psi \to \chi) \to ([\phi]^+\psi \to [\phi]^+\chi)$,

(A_7) $[\phi]^-(\psi \to \chi) \to ([\phi]^-\psi \to [\phi]^-\chi)$,

(A_8) $\psi \to [\phi]^+\langle\phi\rangle^-\psi$,

(A_9) $\psi \to [\phi]^-\langle\phi\rangle^+\psi$,

(A_{10}) $\langle\phi\rangle^+\psi \to [\phi]^+\psi$,

(A_{11}) $\neg\phi \to [\phi]^+\bot$,

(A_{12}) $[\phi]^+\bot \to \neg\phi$,

(A_{13}) $[\top]^+\phi \to \phi$,

(A_{14}) $p \to [\phi]^+p$,

(A_{15}) $\neg p \to [\phi]^+\neg p$,

(A_{16}) $\langle\phi\rangle^+\Box\psi \to \Box[\phi]^+\psi$,

(A_{17}) $\Box[\phi]^+\psi \to [\phi]^+\Box\psi$,

(R_1) $\frac{\phi,\ \phi\to\psi}{\psi}$,

(R_2) $\frac{\phi}{\Box\phi}$,

(R_3) $\frac{\psi}{[\phi]^+\psi}$,

(R_4) $\frac{\psi}{[\phi]^-\psi}$.

We briefly explain the importance of the above axioms and inference rules:

- (A_1) and (R_1) are all we need to prove Lindenbaum Lemma,
- (A_2) and (R_2) are all we need to prove the $\Diamond$-Lemma,
- (A_3)–(A_5) are all we need to prove that $\Box$ gives rise to an equivalence relation between maximal consistent sets of formulas,
- (A_6), (A_7), (R_3) and (R_4) are all we need to prove the $\langle\phi\rangle^{\pm}$-Lemma,
- (A_8) and (A_9) are all we need to prove that $[\phi]^+$ and $[\phi]^-$ give rise to mutually converse relations between maximal consistent sets of formulas,
- (A_{10}) means that announcements are deterministic,
- (A_{11}) and (A_{12}) mean that announcements are executable iff they are true,
- (A_{13}) means, together with (A_{10}), that announcing $\top$ has no effect at all,
- (A_{14}) and (A_{15}) mean that announcements have no effect on the valuation,
- (A_{16}) and (A_{17}) relate what becomes known after an announcement to what was known before it.

As the reader can see, (A_7)–(A_9) are the only axioms explicitly concerning the $[\cdot]^-$ construct. About axioms (A_{10})–(A_{17}), seeing that, apparently, they are less innocent than axioms (A_7)–(A_9), from now on, we will indicate their use.

Lemma 6.1 *The following formulas are in $PAL^{\pm}$:*

- $\phi \wedge [\phi]^+\psi \rightarrow \langle\phi\rangle^+\psi$,
- $p \rightarrow [\phi]^-p$,
- $[\phi]^+p \rightarrow (\phi \rightarrow p)$,
- $[\phi]^+\neg\psi \rightarrow (\phi \rightarrow \neg[\phi]^+\psi)$,
- $[\phi]^+(\psi \vee \chi) \rightarrow [\phi]^+\psi \vee [\phi]^+\chi$.

Proposition 6.2 (Soundness) *Let ϕ be a formula. If $\phi \in PAL^{\pm}$ then $\models \phi$.*

Proof. It suffices to verify that axioms (A_1)–(A_{17}) are valid and inference rules $(R1)$–$(R4)$ are validity-preserving. □

7 Canonical model

A set Γ of formulas is said to be consistent iff for all $n \in \mathbb{N}$ and for all $\phi_1, \ldots, \phi_n \in \Gamma$, $\neg(\phi_1 \wedge \ldots \wedge \phi_n) \notin PAL^{\pm}$. We shall say that a set Γ of formulas is maximal iff for all formulas ϕ, $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. Let U_c be the set of all maximal consistent sets of formulas (with typical members Γ, Δ, etc).

Lemma 7.1 (Lindenbaum Lemma) *Let Γ be a set of formulas. If Γ is consistent then there exists a maximal consistent set Δ of formulas such that $\Gamma \subseteq \Delta$.*

Let $R_\Box$ be the binary relation on U_c such that $\Gamma\ R_\Box\ \Delta$ iff $\Box\Gamma \subseteq \Delta$.

Lemma 7.2 ($\Diamond$-Lemma) *Let ϕ be a formula. Let $\Gamma \in U_c$. If $\Diamond\phi \in \Gamma$ then there exists $\Delta \in U_c$ such that $\Gamma\ R_\Box\ \Delta$ and $\phi \in \Delta$.*

Lemma 7.3 *$R_\Box$ is an equivalence relation on U_c.*

For all formulas ϕ, let $R_{[\phi]^+}$ and $R_{[\phi]^-}$ be the binary relations on U_c such that (i) $\Gamma\ R_{[\phi]^+}\ \Delta$ iff $[\phi]^+\Gamma \subseteq \Delta$ and (ii) $\Gamma\ R_{[\phi]^-}\ \Delta$ iff $[\phi]^-\Gamma \subseteq \Delta$.

Lemma 7.4 ($\langle\phi\rangle^{\pm}$-Lemma) *Let ϕ be a formula. Let $\Gamma \in U_c$.*

- *If $\langle\phi\rangle^+\psi \in \Gamma$ then there exists $\Delta \in U_c$ such that $\Gamma\ R_{[\phi]^+}\ \Delta$ and $\psi \in \Delta$,*
- *if $\langle\phi\rangle^-\psi \in \Gamma$ then there exists $\Delta \in U_c$ such that $\Gamma\ R_{[\phi]^-}\ \Delta$ and $\psi \in \Delta$.*

Lemma 7.5 *Let ϕ be a formula. $R_{[\phi]^+}$ and $R_{[\phi]^-}$ are mutually converse on U_c.*

Lemma 7.6 *Let ϕ be a formula. Let $\Gamma, \Delta \in U_c$. If $\Gamma\ R_{[\phi]^+}\ \Delta$ then $\phi \in \Gamma$ and $[\phi]^+\Gamma = \Delta$.*

Lemma 7.7 *$R_{[\top]^+}$ and $R_{[\top]^-}$ are equal to the identity relation on U_c.*

Let $\equiv$ be the transitive closure of $\bigcup\{R_{[\phi]^+} :\ \phi$ is a formula$\} \cup \bigcup\{R_{[\phi]^-} :\ \phi$ is a formula$\}$.

Lemma 7.8 *$\equiv$ is an equivalence relation on U_c.*

The equivalence class of $\Gamma \in U_c$ modulo $\equiv$ will be simply noted $|\Gamma|$. Let $\equiv_\Box$ be the transitive closure of $R_\Box \cup \equiv$.

Lemma 7.9 *$\equiv_\Box$ is an equivalence relation on U_c. Moreover, $\equiv_\Box$ is coarser than $R_\Box$ and $\equiv$.*

Proposition 7.10 *Let $\Gamma, \Delta, \Lambda, \Theta \in U_c$. Let ϕ be a formula. If $\Box\Gamma \subseteq \Delta$, $[\phi]^+\Gamma \subseteq \Lambda$ and $[\phi]^+\Delta \subseteq \Theta$ then $\Box\Lambda \subseteq \Theta$.*

Proof. Suppose $\Box\Gamma \subseteq \Delta$, $[\phi]^+\Gamma \subseteq \Lambda$ and $[\phi]^+\Delta \subseteq \Theta$. Suppose $\Box\Lambda \not\subseteq \Theta$. Let ψ be a formula such that $\psi \in \Box\Lambda$ and $\psi \notin \Theta$. Hence, $\Box\psi \in \Lambda$. Since $[\phi]^+\Gamma \subseteq \Lambda$, therefore $\langle\phi\rangle^+\Box\psi \in \Gamma$. Using (A_{16}), $\Box[\phi]^+\psi \in \Gamma$. Since $\Box\Gamma \subseteq \Delta$, therefore $[\phi]^+\psi \in \Delta$. Since $[\phi]^+\Delta \subseteq \Theta$, therefore $\psi \in \Theta$: a contradiction. Thus, $\Box\Lambda \subseteq \Theta$. $\Box$

Proposition 7.11 *Let $\Gamma, \Delta, \Lambda \in U_c$. Let ϕ be a formula. If $[\phi]^+\Gamma \subseteq \Delta$ and $\Box\Delta \subseteq \Lambda$ then there exists $\Theta \in U_c$ such that $\Box\Gamma \subseteq \Theta$ and $[\phi]^+\Theta \subseteq \Lambda$.*

Proof. Suppose $[\phi]^+\Gamma \subseteq \Delta$ and $\Box\Delta \subseteq \Lambda$. Suppose $\Box\Gamma \cup \{\langle\phi\rangle^+\varphi' : \varphi' \in \Lambda\}$ is not consistent. Consequently, there exist $\varphi_1, \ldots, \varphi_m \in \Box\Gamma$ and there exist $\varphi'_1, \ldots, \varphi'_n \in \Lambda$ such that $\neg(\varphi_1 \wedge \ldots \wedge \varphi_m \wedge \langle\phi\rangle^+\varphi'_1 \wedge \ldots \wedge \langle\phi\rangle^+\varphi'_n) \in PAL^\pm$. Hence, $\varphi_1 \wedge \ldots \wedge \varphi_m \rightarrow [\phi]^+\neg(\varphi'_1 \wedge \ldots \wedge \varphi'_n) \in PAL^\pm$. Thus, $\Box(\varphi_1 \wedge \ldots \wedge \varphi_m) \rightarrow \Box[\phi]^+\neg(\varphi'_1 \wedge \ldots \wedge \varphi'_n) \in PAL^\pm$. Since $\varphi_1, \ldots, \varphi_m \in \Box\Gamma$, therefore $\Box(\varphi_1 \wedge \ldots \wedge \varphi_m) \in \Gamma$. Since $\Box(\varphi_1 \wedge \ldots \wedge \varphi_m) \rightarrow \Box[\phi]^+\neg(\varphi'_1 \wedge \ldots \wedge \varphi'_n) \in PAL^\pm$, therefore $\Box[\phi]^+\neg(\varphi'_1 \wedge \ldots \wedge \varphi'_n) \in \Gamma$. Using (A_{17}), $[\phi]^+\Box\neg(\varphi'_1 \wedge \ldots \wedge \varphi'_n) \in \Gamma$. Since $[\phi]^+\Gamma \subseteq \Delta$, therefore $\Box\neg(\varphi'_1 \wedge \ldots \wedge \varphi'_n) \in \Delta$. Since $\Box\Delta \subseteq \Lambda$, therefore $\neg(\varphi'_1 \wedge \ldots \wedge \varphi'_n) \in \Lambda$. Consequently, $\varphi'_1 \notin \Lambda$ or ... or $\varphi'_n \notin \Lambda$: a contradiction. Hence, $\Box\Gamma \cup \{\langle\phi\rangle^+\varphi' : \varphi' \in \Lambda\}$ is consistent. Let $\Theta \in U_c$ be such that $\Box\Gamma \cup \{\langle\phi\rangle^+\varphi' : \varphi' \in \Lambda\} \subseteq \Theta$. Thus, $\Box\Gamma \subseteq \Theta$ and $[\phi]^+\Theta \subseteq \Lambda$. $\Box$

For all $\Gamma_0 \in U_c$, let $\mathcal{M}_{\Gamma_0} = (W_{\Gamma_0}, X_{\Gamma_0}, V_{\Gamma_0})$ be the model such that $W_{\Gamma_0} = \{|\Gamma| : \Gamma_0 \equiv_\Box \Gamma\}$, $X_{\Gamma_0} = \{S_\Box(\Gamma) : \Gamma_0 \equiv_\Box \Gamma\}$ where $S_\Box(\Gamma) = \{|\Delta| : \Gamma\ R_\Box\ \Delta\}$ and $V_{\Gamma_0}(p) = \{|\Gamma| : \Gamma_0 \equiv_\Box \Gamma \text{ and } p \in \Gamma\}$. For all $\Gamma_0 \in U_c$, $\mathcal{M}_{\Gamma_0}$ will be called Γ_0-canonical model. Each maximal consistent set of formulas equivalent with Γ_0 modulo $\equiv_\Box$ should be seen as a moment in the history of a world. If two of them are different but equivalent modulo $\equiv$, this means that they correspond to different moments in the history of the same world. For this reason, $\mathcal{M}_{\Gamma_0}$-worlds are equivalence classes modulo $\equiv$ of maximal consistent sets of formulas equivalent with Γ_0 modulo $\equiv_\Box$. As for $\mathcal{M}_{\Gamma_0}$-steps, each of them is determined by a moment in the history of a world and consists of the set of all $\mathcal{M}_{\Gamma_0}$-worlds that are equivalent with this moment modulo $R_\Box$. The thing is that one should understand $R_\Box$ as an equivalence relation of contemporaneity between moments. Concerning the $\mathcal{M}_{\Gamma_0}$-valuation, as expected, it associates to each atomic formula the set of all $\mathcal{M}_{\Gamma_0}$-worlds that contain a moment containing the atomic formula.

8 Truth Lemma

For an arbitrary $\Gamma_0 \in U_c$, let P be the set of all formulas ϕ such that for all $\Gamma, \Delta \in U_c$, if $\Gamma_0 \equiv_\Box \Gamma$, $\Gamma_0 \equiv_\Box \Delta$ and $|\Gamma| \in S_\Box(\Delta)$ then the 3 following conditions C_1–C_3 are equivalent:

(C_1) $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models \phi$,

(C_2) there exists $\Lambda \in U_c$ such that $\Gamma \equiv \Lambda$, $\Delta\ R_\Box\ \Lambda$ and $\phi \in \Lambda$,

(C_3) for all $\Lambda' \in U_c$, if $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$ then $\phi \in \Lambda'$.

Proposition 8.1 *Let* $\psi \in P$*. Let* $\Delta \in U_c$*. Let* $T = \{|\Pi| \in S_\Box(\Delta) : \mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta)) \models \psi\}$*. Let* $\Sigma, \Lambda \in U_c$*. If* $\Delta\ R_\Box\ \Sigma$*,* $\psi \in \Sigma$ *and* $[\psi]^+\Sigma = \Lambda$ *then* $T = S_\Box(\Lambda)$*.*

Proof. Suppose $\Delta\ R_\Box\ \Sigma$, $\psi \in \Sigma$ and $[\psi]^+\Sigma = \Lambda$. Let $\Pi \in U_c$.
Suppose $|\Pi| \in T$. Hence, $|\Pi| \in S_\Box(\Delta)$ and $\mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta)) \models \psi$. Let $\Pi' \in U_c$ be such that $\Pi \equiv \Pi'$, $\Delta\ R_\Box\ \Pi'$ and $\psi \in \Pi'$. Such $\Pi' \in U_c$ exists because $\psi \in P$. Suppose $\Box\Lambda \not\subseteq [\psi]^+\Pi'$. Let φ be a formula such that $\varphi \in \Box\Lambda$ and $\varphi \notin [\psi]^+\Pi'$. Thus, $\Box\varphi \in \Lambda$ and $[\psi]^+\varphi \notin \Pi'$. Since $[\psi]^+\Sigma = \Lambda$, therefore $\Box\varphi \in [\psi]^+\Sigma$. Consequently, $[\psi]^+\Box\varphi \in \Sigma$. By Lemma 6.1, since $\psi \in \Sigma$, therefore $\langle\psi\rangle^+\Box\varphi \in \Sigma$. Using (A_{16}), $\Box[\psi]^+\varphi \in \Sigma$. Since $\Delta\ R_\Box\ \Sigma$ and $\Delta\ R_\Box\ \Pi'$, therefore $\Sigma\ R_\Box\ \Pi'$. Since $\Box[\psi]^+\varphi \in \Sigma$, therefore $[\psi]^+\varphi \in \Pi'$: a contradiction. Hence, $\Box\Lambda \subseteq [\psi]^+\Pi'$. Thus, $\Lambda\ R_\Box\ [\psi]^+\Pi'$. Since $\Pi \equiv \Pi'$, therefore $\Pi \equiv [\psi]^+\Pi'$. Since $\Lambda\ R_\Box\ [\psi]^+\Pi'$, therefore $|\Pi| \in S_\Box(\Lambda)$.
Suppose $|\Pi| \in S_\Box(\Lambda)$. Let $\Pi' \in U_c$ be such that $\Pi \equiv \Pi'$ and $\Lambda\ R_\Box\ \Pi'$. By Proposition 7.11, since $[\psi]^+\Sigma = \Lambda$, let $\Theta' \in U_c$ be such that $\Sigma\ R_\Box\ \Theta'$ and $\Theta'\ R_{[\psi]^+}\ \Pi'$. Since $\Delta\ R_\Box\ \Sigma$, therefore $\Delta\ R_\Box\ \Theta'$. Since $\Pi \equiv \Pi'$ and $\Theta'\ R_{[\psi]^+}\ \Pi'$, therefore $\Pi \equiv \Theta'$. Since $\Delta\ R_\Box\ \Theta'$, therefore $|\Pi| \in S_\Box(\Delta)$. By Lemma 7.6, since $\Theta'\ R_{[\psi]^+}\ \Pi'$, therefore $\psi \in \Theta'$. Since $\Pi \equiv \Theta'$, $\Delta\ R_\Box\ \Theta'$ and $\psi \in P$, therefore $\mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta)) \models \psi$. Consequently, $|\Pi| \in T$. □

Proposition 8.2 (Truth Lemma) *For all formulas* ϕ*,* $\phi \in P$*.*

Proof. The proof is done by induction, based on the function $size(\cdot)$ defined in Section 2. Let ϕ be a formula such that for all formulas ψ, if $size(\psi) < size(\phi)$ then $\psi \in P$. We demonstrate $\phi \in P$. Let $\Gamma, \Delta \in U_c$ be such that $\Gamma_0 \equiv_\Box \Gamma$, $\Gamma_0 \equiv_\Box \Delta$ and $|\Gamma| \in S_\Box(\Delta)$. We demonstrate the 3 above conditions C_1–C_3 are equivalent. Let $\Theta \in U_c$ be such that $\Gamma \equiv \Theta$ and $\Delta\ R_\Box\ \Theta$. We have to consider the following 7 cases: $\phi = p$, $\phi = \bot$, $\phi = \neg\psi$, $\phi = \psi \vee \chi$, $\phi = \Box\psi$, $\phi = [\psi]^+\chi$ and $\phi = [\psi]^-\chi$. For the sake of brevity, we only present the most difficult of them, the case $\phi = [\psi]^-\chi$. The cases $\phi = \Box\psi$ and $\phi = [\psi]^+\chi$ are presented in the Annex.
Case $\phi = [\psi]^-\chi$**.** Since $size(\psi) < size(\phi)$ and $size(\chi) < size(\phi)$, therefore $\psi \in P$ and $\chi \in P$.
$(C_1 \Rightarrow C_2)$. Suppose $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models [\psi]^-\chi$. Suppose $[\psi]^-\chi \notin \Theta$. By Lemma 7.4, let $\Lambda \in U_c$ be such that $\Theta\ R_{[\psi]^-}\ \Lambda$ and $\chi \notin \Lambda$. By Lemma 7.6, $\psi \in \Lambda$ and $[\psi]^+\Lambda = \Theta$. Since $\Gamma \equiv \Theta$ and $\Theta\ R_{[\psi]^-}\ \Lambda$, therefore $\Gamma \equiv \Lambda$. Let $T = \{|\Pi| \in S_\Box(\Lambda) : \mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Lambda)) \models \psi\}$. By Proposition 8.1, since $\psi \in \Lambda$ and $[\psi]^+\Lambda = \Theta$, therefore $T = S_\Box(\Theta)$. Since $\Delta\ R_\Box\ \Theta$, therefore $T = S_\Box(\Delta)$. Since $\Gamma \equiv \Lambda$, therefore $|\Gamma| \in S_\Box(\Lambda)$. Since $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models [\psi]^-\chi$ and $T = S_\Box(\Delta)$, therefore $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Lambda)) \models \chi$. Since $\Gamma \equiv \Lambda$ and $\chi \in P$, therefore $\chi \in \Lambda$: a contradiction. Thus, $[\psi]^-\chi \in \Theta$.

$(C_2 \Rightarrow C_3)$. Suppose $\Lambda \in U_c$ is such that $\Gamma \equiv \Lambda$, $\Delta\ R_\Box\ \Lambda$ and $[\psi]^-\chi \in \Lambda$. Let $\Lambda' \in U_c$ be such that $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$. Suppose $[\psi]^-\chi \notin \Lambda'$. By Lemma 7.4, let $\Lambda'' \in U_c$ be such that $\Lambda'\ R_{[\psi]^-}\ \Lambda''$ and $\chi \notin \Lambda''$. By Lemma 7.6, $\psi \in \Lambda''$ and $[\psi]^+\Lambda'' = \Lambda'$. Since $\Delta\ R_\Box\ \Lambda$ and $\Delta\ R_\Box\ \Lambda'$, therefore $\Lambda'\ R_\Box\ \Lambda$. Since $\Lambda''\ R_{[\psi]^+}\ \Lambda'$, therefore by Proposition 7.11, let $\Lambda''' \in U_c$ be such that $\Lambda''\ R_\Box\ \Lambda'''$ and $\Lambda'''\ R_{[\psi]^+}\ \Lambda$. Since $[\psi]^-\chi \in \Lambda$, therefore $\chi \in \Lambda'''$. Since $\Gamma \equiv \Lambda$, $\Gamma \equiv \Lambda'$, $\Lambda'\ R_{[\psi]^-}\ \Lambda''$ and $\Lambda'''\ R_{[\psi]^+}\ \Lambda$, therefore $\Lambda''' \equiv \Lambda''$. Since $\Lambda''\ R_\Box\ \Lambda'''$, $\chi \in \Lambda'''$ and $\chi \in P$, therefore $\chi \in \Lambda''$: a contradiction. Hence, $[\psi]^-\chi \in \Lambda'$.

$(C_3 \Rightarrow C_1)$. Suppose for all $\Lambda' \in U_c$, if $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$ then $[\psi]^-\chi \in \Lambda'$. Suppose $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \not\models [\psi]^-\chi$. Hence, there exists $T \in X_{\Gamma_0}$ such that $|\Gamma| \in T$, $S_\Box(\Delta) = \{|\Pi| \in T : \mathcal{M}_{\Gamma_0}, (|\Pi|, T \models \psi\}$ and $\mathcal{M}_{\Gamma_0}, (|\Gamma|, T) \not\models \chi$. Let $\Delta' \in U_c$ be such that $\Gamma_0 \equiv_\Box \Delta'$ and $T = S_\Box(\Delta')$. Since $\Gamma \equiv \Theta$, $\Delta\ R_\Box\ \Theta$ and for all $\Lambda' \in U_c$, if $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$ then $[\psi]^-\chi \in \Lambda'$, therefore $[\psi]^-\chi \in \Theta$. Since $|\Gamma| \in T$, $S_\Box(\Delta) = \{|\Pi| \in T : \mathcal{M}_{\Gamma_0}, (|\Pi|, T \models \psi\}$, $\mathcal{M}_{\Gamma_0}, (|\Gamma|, T) \not\models \chi$ and $T = S_\Box(\Delta')$, therefore $|\Gamma| \in S_\Box(\Delta')$, $S_\Box(\Delta) = \{|\Pi| \in S_\Box(\Delta') : \mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta') \models \psi\}$ and $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta')) \not\models \chi$. Let $\Theta' \in U_c$ be such that $\Gamma \equiv \Theta'$, $\Delta'\ R_\Box\ \Theta'$ and $\chi \notin \Theta'$. Such $\Theta' \in U_c$ exists because $\chi \in P$. Since $|\Gamma| \in S_\Box(\Delta)$ and $S_\Box(\Delta) = \{|\Pi| \in S_\Box(\Delta') : \mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta') \models \psi\}$, therefore $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta') \models \psi$. Since $\Gamma \equiv \Theta'$, $\Delta'\ R_\Box\ \Theta'$ and $\psi \in P$, therefore $\psi \in \Theta'$. Let $R_\Box^\psi(\Delta')$ be the set of all $\Pi' \in R_\Box(\Delta')$ such that $\psi \in \Pi'$. Since $\Delta'\ R_\Box\ \Theta'$ and $\psi \in \Theta'$, therefore $\Theta' \in R_\Box^\psi(\Delta')$. Since $\Gamma \equiv \Theta$ and $\Gamma \equiv \Theta'$, therefore $\Theta \equiv \Theta'$. Let Tri be the set of all triples of the form (d, m, φ) where $d \in \mathbb{N}$, $m \in \mathbb{N}$ and φ is a formula. Let Q be the set of all $(d, m, \varphi) \in Tri$ such that for all formulas $\varphi_1, \ldots, \varphi_m$, if $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_m)) + deg(\varphi) \leq d$ then for all $s_1, \ldots, s_m \in \{+, -\}$, for all $\Pi' \in R_\Box^\psi(\Delta')$ and for all $\Pi \in R_\Box(\Delta)$, if $\Pi' \equiv \Pi$ then the 2 following conditions hold:

(D_1) for all $\Pi'_1, \ldots, \Pi'_m \in U_c$, if $[\psi]^+\Pi'\ R_{[\varphi_1]^{s_1}}\ \Pi'_1$, $\Pi'_1\ R_{[\varphi_2]^{s_2}}\ \Pi'_2$, ..., $\Pi'_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi'_m$ then there exist $\Pi_1, \ldots, \Pi_m \in U_c$ such that $\Pi\ R_{[\varphi_1]^{s_1}}\ \Pi_1$, $\Pi_1\ R_{[\varphi_2]^{s_2}}\ \Pi_2$, ..., $\Pi_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi_m$ and if $\varphi \in \Pi'_m$ then $\varphi \in \Pi_m$,

(D_2) for all $\Pi_1, \ldots, \Pi_m \in U_c$, if $\Pi\ R_{[\varphi_1]^{s_1}}\ \Pi_1$, $\Pi_1\ R_{[\varphi_2]^{s_2}}\ \Pi_2$, ..., $\Pi_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi_m$ then there exist $\Pi'_1, \ldots, \Pi'_m \in U_c$ such that $[\psi]^+\Pi'\ R_{[\varphi_1]^{s_1}}\ \Pi'_1$, $\Pi'_1\ R_{[\varphi_2]^{s_2}}\ \Pi'_2$, ..., $\Pi'_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi'_m$ and if $\varphi \in \Pi_m$ then $\varphi \in \Pi'_m$.

In the above definition, we use the product $deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_m)$ of the degrees of the formulas $\varphi_1, \ldots, \varphi_m$. Since m may be equal to 0, we will consider that in this case, such product is equal to 2. The following claims illustrate the interest to consider the set Tri and its subset Q.

Claim (a)**:**

(i) For all $\Pi' \in R_\Box^\psi(\Delta')$, there exists $\Pi \in R_\Box(\Delta)$ such that $\Pi' \equiv \Pi$,

(ii) for all $\Pi \in R_\Box(\Delta)$, there exists $\Pi' \in R_\Box^\psi(\Delta')$ such that $\Pi' \equiv \Pi$.

Claim (b)**:** If $Q = Tri$ then $[\psi]^+\Theta' \subseteq \Theta$.

Claim (c)**:** $Q = Tri$.

Claim (a) clearly shows the tight relationships between $R^{\psi}_{\Box}(\Delta')$ and $R_{\Box}(\Delta)$. It is only used in the proof of Claim (c). Now, by Claims (b) and (c), $[\psi]^+\Theta' \subseteq \Theta$. Hence, $[\psi]^-\Theta \subseteq \Theta'$. Since $[\psi]^-\chi \in \Theta$, therefore $\chi \in \Theta'$: a contradiction.□

Lemma 8.3 *For all* $\Gamma_0 \in U_c$*,* $\mathcal{M}_{\Gamma_0}$ *is free.*

Proposition 8.4 (Completeness) *Let* ϕ *be a formula. If* $\models \phi$ *then* $\phi \in PAL^{\pm}$*.*

Proof. Suppose $\models \phi$ and $\phi \not\in PAL^{\pm}$. Let $\Gamma_0 \in U_c$ be such that $\phi \not\in \Gamma_0$. By Proposition 8.2, $\mathcal{M}_{\Gamma_0}, (|\Gamma_0|, S_{\Box}(\Gamma_0)) \not\models \phi$. Moreover, by Lemma 8.3, $\mathcal{M}_{\Gamma_0}$ is free. Thus, $\not\models \phi$: a contradiction. □

9 Maximal ignorance

The model of maximal ignorance is the triple $\mathcal{M}_0 = (W_0, X_0, V_0)$ where $W_0 = 2^{VAR}$, $X_0 = 2^{2^{VAR}} \setminus \{\emptyset\}$ and V_0 is the function associating to each $p \in VAR$ the subset $V_0(p)$ of W_0 defined as follows: $x \in V_0(p)$ iff $p \in x$. In $\mathcal{M}_0$, each subset x of VAR represents an epistemic alternative for the real world and each non-empty set S of subsets of VAR contains the real world together with its epistemic alternatives at some moment of their history. Moreover, each world-step pair (x, S) such that $x \in S$ determines the real world x and the current restriction of the model containing the real world together with its epistemic alternatives.

Lemma 9.1 $\mathcal{M}_0$ *is free.*

From now on in this section, we will say that a formula ϕ is 0-valid, in symbols $\models_0 \phi$, if ϕ is globally true in $\mathcal{M}_0$. In this section, we investigate the set of all 0-valid formulas. This set is of special interest as we recover some of the original intuitions for the logic of "what is true before an announcement", for example the validity of the formula $\Box p \to \langle p\rangle^-\Diamond\neg p$ mentioned in the introduction.

Proposition 9.2 *Let* ϕ *be a formula. If* ϕ *is* $\{[\cdot]^+, [\cdot]^-\}$*-free then* $\models_0 \phi$ *iff* $\phi \in S5$*.*

Proof. By [10, Pages 29 and 30]. □

Let $k \in \mathbb{N}$.

Lemma 9.3 *Let* A *and* B *be* k*-steps. If* $A \subseteq B$ *then the formulas* $\nabla B \to [\bigvee A]^+\nabla A$ *and* $\nabla A \to \langle\bigvee A\rangle^-\nabla B$ *are* 0*-valid.*

Proposition 9.4 *Let* ϕ *be a* k*-formula. If* ϕ *is* $\{[\cdot]^+, [\cdot]^-\}$*-free then there exists a family* $\{(\alpha_1, A_1), \ldots, (\alpha_m, A_m)\}$ *of* k*-tips such that* $\models_0 \phi \leftrightarrow \bigvee\{\alpha_i \wedge \nabla A_i : 1 \leq i \leq m\}$*.*

For all $\mathcal{M}_0$-worlds x, let $f_k(x)$ be the unique k-world V-agreeing with x. For all $\mathcal{M}_0$-steps S, let $F_k(S) = \{f_k(x) : x \in S\}$ be the unique k-step consisting of all k-worlds V-agreeing with an $\mathcal{M}_0$-world in S. Obviously, for all $\mathcal{M}_0$-tips (x, S), the pair $(f_k(x), F_k(S))$ is a k-tip. Moreover, $f_k(x)$ is a finite conjunction of literals over $p_1, \ldots, p_k$ and $F_k(S)$ is a non-empty finite set of finite conjunctions of literals over $p_1, \ldots, p_k$.

Lemma 9.5 *For all $\mathcal{M}_0$-tips (x, S) and for all k-tips (α, A), $\mathcal{M}_0, (x, S) \models \alpha \wedge \nabla A$ iff $f_k(x) = \alpha$ and $F_k(S) = A$.*

Lemma 9.6 *For all k-formulas ϕ and for all $\mathcal{M}_0$-tips $(x, S), (y, T)$, if $f_k(x) = f_k(y)$ and $F_k(S) = F_k(T)$ then $\mathcal{M}_0, (x, S) \models \phi$ iff $\mathcal{M}_0, (y, T) \models \phi$.*

Proposition 9.7 *Let ϕ be a k-formula. The formula $\phi \leftrightarrow \bigvee\{f_k(x) \wedge \nabla F_k(S) : x \in W_0 \; \& \; S \in X_0 \; \& \; x \in S \; \& \; \mathcal{M}_0, (x, S) \models \phi\}$ is 0-valid.*

Proof. Let (y, T) be an $\mathcal{M}_0$-tip.
Suppose $\mathcal{M}_0, (y, T) \models \phi$. By Lemma 9.5, $\mathcal{M}_0, (y, T) \models f_k(y) \wedge \nabla F_k(T)$. Since $\mathcal{M}_0, (y, T) \models \phi$, therefore $\mathcal{M}_0, (y, T) \models \bigvee\{f_k(x) \wedge \nabla F_k(S) : x \in W_0 \; \& \; S \in X_0 \; \& \; x \in S \; \& \; \mathcal{M}_0, (x, S) \models \phi\}$.
Suppose $\mathcal{M}_0, (y, T) \models \bigvee\{f_k(x) \wedge \nabla F_k(S) : x \in W_0 \; \& \; S \in X_0 \; \& \; x \in S \; \& \; \mathcal{M}_0, (x, S) \models \phi\}$. Let $x \in W_0$ and $S \in X_0$ be such that $x \in S$, $\mathcal{M}_0, (x, S) \models \phi$ and $\mathcal{M}_0, (y, T) \models f_k(x) \wedge \nabla F_k(S)$. Hence, by Lemma 9.5, $f_k(y) = f_k(x)$ and $F_k(T) = F_k(S)$. Since $\mathcal{M}_0, (x, S) \models \phi$, therefore by Lemma 9.6, $\mathcal{M}_0, (y, T) \models \phi$. □

Proposition 9.8 *Let ϕ be a k-formula. There exists a $\{[\cdot]^+, [\cdot]^-\}$-free formula ψ such that $\models_0 \phi \leftrightarrow \psi$.*

Proof. By Proposition 9.7. □

Proposition 9.8 says that the constructs $[\cdot]^+$ and $[\cdot]^-$ can be eliminated from the language as far as 0-validity is concerned. It does not say how, though. To be able to say how, it suffices to be able to determine in particular which $\{[\cdot]^+, [\cdot]^-\}$-free formulas are 0-equivalent to $\langle\phi\rangle^+\psi$ and $\langle\phi\rangle^-\psi$ when the formulas ϕ and ψ are already $\{[\cdot]^+, [\cdot]^-\}$-free.

Proposition 9.9 *Let $\{(\alpha_1, A_1), \ldots, (\alpha_m, A_m)\}$ be a family of k-tips and (β, B) be a k-tip. Let $\phi = \bigvee\{\alpha_i \wedge \nabla A_i : 1 \leq i \leq m\}$ and $\psi = \beta \wedge \nabla B$. If $B_\phi = \{\alpha_i : 1 \leq i \leq m \; \& \; A_i = B\}$ then the formulas $\langle\phi\rangle^+\psi \leftrightarrow \beta \wedge \nabla_\phi B$ and $\langle\phi\rangle^-\psi \leftrightarrow \beta \wedge \nabla B_\phi$ are 0-valid.*

By Propositions 9.4 and 9.9, one can easily design a procedure computing for any given input formula a 0-equivalent $\{[\cdot]^+, [\cdot]^-\}$-free formula. For instance, the formula $\langle p\rangle^-\top$ is 0-equivalent to the $\{[\cdot]^+, [\cdot]^-\}$-free formula $\Box p$.

10 Conclusion

There are several ways to continue this research.
Firstly, there are computability issues. Within the context of the model of maximal ignorance, using the fact that for each $k \in \mathbb{N}$, there exist exactly

2^k k-worlds and there exist exactly $2^{2^k} - 1$ k-steps, one readily sees that the validity problem is decidable, although its exact complexity is still unknown. Within the context of the class of all free models, the computability of the validity problem is still open. Other computability issues are related to the problem consisting of given a formula ϕ, either to determine if there exists a formula ψ such that $\langle\psi\rangle^+\phi$ is valid, or to determine if there exists a formula ψ such that $\langle\psi\rangle^-\phi$ is valid.
Secondly, there are multi-agent issues. There exist already many multi-agent variants of subset space logic. Within the context of our logic of knowledge with public announcements and converse public announcements, we did not manage to find its acceptable multi-agent variant.
Thirdly, there are introspection issues. In our setting, the agent is both positively and negatively introspective. Suppose the agent is non negatively introspective. This implies that we have to get rid of axiom (A_4). But this also implies that the binary relation $R_\Box$ defined in Section 7 is no more an equivalence relation on U_c. And we did not manage to completely axiomatize the corresponding logic of knowledge with public announcements and converse public announcements. Remark that subset space logics of a merely positively introspective agent do not seem to exist.
Fourthly, there is the issue of the extension with the constructs $\blacksquare^+$ and $\blacksquare^-$ considered in Section 5 and corresponding to the effort modality of Subset Space Logic [5,13] and the converse effort modality introduced by Heinemann [9]. This extension is of great interest as, in our setting, $\blacksquare^+$ is like an arbitrary announcement modality [2], and thus $\blacksquare^-$ an "arbitrary before the announcement" modality. How to completely axiomatize this extension is still open.
Fifthly, there are characterization issues. For example, the characterization of the set of all pairs (ϕ, ψ) of formulas such that $[\phi]^+\psi$ is valid and the characterization of the set of all pairs (ϕ, ψ) of formulas such that $[\phi]^-\psi$ is valid.

Acknowledgements

We would like to thank the referees for the feedback we have obtained from them. Philippe Balbiani and Hans van Ditmarsch were partially supported by the ERC project EPS 313360. This work is also supported by the "Agence nationale de la recherche" (contract ANR-11-BS02-011). Special thanks are due to the members of the DynRes project for their extensive remarks concerning a preliminary version of the present paper.

References

[1] Aucher, G., Herzig, A.: *From DEL to EDL: exploring the power of converse events.* In: *Proceedings of ECSQARU 2007.* Springer (2007) 199–209.

[2] Balbiani, P., Baltag, A., van Ditmarsch, H., Herzig, A., Hoshi, T., de Lima, T.: *"Knowable" as "known after an announcement".* The Review of Symbolic Logic **1** (2008) 305–334.

[3] Baltag, A., Moss, L.: *Logics for epistemic programs.* Synthese **139** (2004) 165–224.

[4] Bozzelli, L., van Ditmarsch, H., French, T., Hales, J., Pinchinat, S.: *Refinement modal logic.* Information and Computation **239** (2014) 303–339.
[5] Dabrowski, A., Moss, L., Parikh, R.: *Topological reasoning and the logic of knowledge.* Annals of Pure and Applied Logic **78** (1996) 73–110.
[6] Van Ditmarsch, H., van Eijck, J., Wu, W.: *One hundred prisoners and a lightbulb — logic and computation.* In: *Proceedings of KR 2010.* AAAI (2010) 90–100.
[7] Van Ditmarsch, H., van der Hoek, W., Kooi, B.: *Dynamic Epistemic Logic.* Springer (2007).
[8] Van Ditmarsch, H., Ruan, J., Verbrugge, R.: *Sum and product in dynamic epistemic logic.* Journal of Logic and Computation **18** (2007) 563–588.
[9] Heinemann, B.: *Including the past in topologic.* In: *Proceedings of LFCS 2007.* Springer (2007) 269–283.
[10] Van der Hoek, W., Meyer, J.-J.: *Epistemic Logic for AI and Computer Science.* Cambridge University Press (1995).
[11] Hoshi, T., Yap, A.: *Dynamic epistemic logic with branching temporal structures.* Synthese (2009) 259–281.
[12] Lutz, C.: *Complexity and succinctness of public announcement logic.* In: *Proceedings of AAMAS 2006.* ACM (2006) 137–143.
[13] Parikh, R., Moss, L., Steinsvold, C.: *Topology and epistemic logic.* In: *Handbook of Spatial Logics.* Springer (2007) 299–341.
[14] Parikh, R., Ramanujam, R.: *A knowledge based semantics of messages.* Journal of Logic, Language and Information **12** (2003) 453–467.
[15] Plaza, J.: *Logics of public communications.* Synthese **158** (2007) 165–179.
[16] Renne, B., Sack, J., Yap, A.: *Dynamic epistemic temporal logic.* In: *Proceedings of LORI 2009.* Springer (2009) 263–277.
[17] Renne, B., Sack, J., Yap, A.: *Logics of temporal-epistemic actions.* Synthese **193** 813–849.
[18] Sack, J.: *Temporal languages for epistemic programs.* Journal of Logic, Language and Information (2008) 183–216.
[19] Wáng, Y., Ågotnes, T.: *Subset space public announcement logic.* In: *Proceedings of ICLA 2013.* Springer (2013) 245–257.
[20] Yap, A.: *Dynamic epistemic logic and temporal modality.* In: *Dynamic Formal Epistemology.* Springer (2011) 33–50.

Annex

Proof of Proposition 8.2: Case $\phi = \Box\psi$. Since $size(\psi) < size(\phi)$, therefore $\psi \in P$.
$(C_1 \Rightarrow C_2)$. Suppose $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models \Box\psi$. Suppose $\Box\psi \notin \Theta$. By Lemma 7.2, let $\Lambda \in U_c$ be such that $\Theta\ R_\Box\ \Lambda$ and $\psi \notin \Lambda$. Since $\Delta\ R_\Box\ \Theta$, therefore $\Delta\ R_\Box\ \Lambda$. Hence, $|\Lambda| \in S_\Box(\Delta)$. Since $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models \Box\psi$, therefore $\mathcal{M}_{\Gamma_0}, (|\Lambda|, S_\Box(\Delta)) \models \psi$. Since $\psi \in P$ and $\Delta\ R_\Box\ \Lambda$, therefore $\psi \in \Lambda$: a contradiction. Thus, $\Box\psi \in \Theta$.
$(C_2 \Rightarrow C_3)$. Suppose $\Lambda \in U_c$ is such that $\Gamma \equiv \Lambda$, $\Delta\ R_\Box\ \Lambda$ and $\Box\psi \in \Lambda$. Let $\Lambda' \in U_c$ be such that $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$. Suppose $\Box\psi \notin \Lambda'$. By Lemma 7.2, let $\Pi \in U_c$ be such that $\Lambda'\ R_\Box\ \Pi$ and $\psi \notin \Pi$. Since $\Delta\ R_\Box\ \Lambda$ and $\Delta\ R_\Box\ \Lambda'$, therefore $\Lambda\ R_\Box\ \Pi$. Since $\Box\psi \in \Lambda$, therefore $\psi \in \Pi$: a contradiction. Hence, $\Box\psi \in \Lambda'$.
$(C_3 \Rightarrow C_1)$. Suppose for all $\Lambda' \in U_c$, if $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$ then $\Box\psi \in \Lambda'$. Since $\Gamma \equiv \Theta$ and $\Delta\ R_\Box\ \Theta$, therefore $\Box\psi \in \Theta$. Suppose $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \not\models \Box\psi$. Let $|\Lambda| \in S_\Box(\Delta)$ be such that $\mathcal{M}_{\Gamma_0}, (|\Lambda|, S_\Box(\Delta)) \not\models \psi$. Let $\Pi \in U_c$ be such that $\Lambda \equiv \Pi$, $\Delta\ R_\Box\ \Pi$ and $\psi \notin \Pi$. Such $\Pi \in U_c$ exists because

$\psi \in P$. Since $\Delta\ R_\Box\ \Theta$, therefore $\Theta\ R_\Box\ \Pi$. Since $\Box\psi \in \Theta$, therefore $\psi \in \Pi$: a contradiction. Hence, $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models \Box\psi$.

Proof of Proposition 8.2: Case $\phi = [\psi]^+\chi$. Since $size(\psi) < size(\phi)$ and $size(\chi) < size(\phi)$, therefore $\psi \in P$ and $\chi \in P$.
$(C_1 \Rightarrow C_2)$. Suppose $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models [\psi]^+\chi$. Suppose $[\psi]^+\chi \notin \Theta$. By Lemma 7.4, let $\Lambda \in U_c$ be such that $\Theta\ R_{[\psi]^+}\ \Lambda$ and $\chi \notin \Lambda$. By Lemma 7.6, $\psi \in \Theta$ and $[\psi]^+\Theta = \Lambda$. Let $T = \{|\Pi| \in S_\Box(\Delta) : \mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta)) \models \psi\}$. By Proposition 8.1, since $\Delta\ R_\Box\ \Theta$, $\psi \in \Theta$ and $[\psi]^+\Theta = \Lambda$, therefore $T = S_\Box(\Lambda)$. Hence, $T \in X_{\Gamma_0}$. Since $\Gamma \equiv \Theta$, $\Delta\ R_\Box\ \Theta$, $\psi \in \Theta$ and $\psi \in P$, therefore $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models \psi$. Thus, $|\Gamma| \in T$. Since $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models [\psi]^+\chi$, therefore $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Lambda)) \models \chi$. Since $\Gamma \equiv \Theta$ and $\Theta\ R_{[\psi]^+}\ \Lambda$, therefore $\Gamma \equiv \Lambda$. Since $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Lambda)) \models \chi$ and $\chi \in P$, therefore $\chi \in \Lambda$: a contradiction. Consequently, $[\psi]^+\chi \in \Theta$.
$(C_2 \Rightarrow C_3)$. Suppose $\Lambda \in U_c$ is such that $\Gamma \equiv \Lambda$, $\Delta\ R_\Box\ \Lambda$ and $[\psi]^+\chi \in \Lambda$. Let $\Lambda' \in U_c$ be such that $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$. Suppose $[\psi]^+\chi \notin \Lambda'$. By Lemma 7.4, let $\Lambda'' \in U_c$ be such that $\Lambda'\ R_{[\psi]^+}\ \Lambda''$ and $\chi \notin \Lambda''$. By Lemma 7.6, $\psi \in \Lambda'$ and $[\psi]^+\Lambda' = \Lambda''$. Since $\Gamma \equiv \Lambda'$, $\Delta\ R_\Box\ \Lambda'$, $\psi \in P$, $\Gamma \equiv \Lambda$ and $\Delta\ R_\Box\ \Lambda$, therefore $\psi \in \Lambda$. By Lemma 6.1, since $[\psi]^+\chi \in \Lambda$, therefore $\langle\psi\rangle^+\chi \in \Lambda$. By Lemma 7.4, let $\Lambda''' \in U_c$ be such that $\Lambda\ R_{[\psi]^+}\ \Lambda'''$ and $\chi \in \Lambda'''$. By Lemma 7.6, $[\psi]^+\Lambda = \Lambda'''$. Since $\Delta\ R_\Box\ \Lambda$ and $\Delta\ R_\Box\ \Lambda'$, therefore $\Lambda\ R_\Box\ \Lambda'$. By Proposition 7.10, since $[\psi]^+\Lambda = \Lambda'''$ and $[\psi]^+\Lambda' = \Lambda''$, therefore $\Lambda'''\ R_\Box\ \Lambda''$. Since $\Gamma \equiv \Lambda$ and $[\psi]^+\Lambda = \Lambda'''$, therefore $\Gamma \equiv \Lambda'''$. Since $\Gamma \equiv \Lambda'$ and $[\psi]^+\Lambda' = \Lambda''$, therefore $\Gamma \equiv \Lambda''$. Since $\Gamma \equiv \Lambda'''$, $\chi \in \Lambda'''$, $\chi \in P$ and $\Lambda'''\ R_\Box\ \Lambda''$, therefore $\chi \in \Lambda''$: a contradiction. Hence, $[\psi]^+\chi \in \Lambda'$.
$(C_3 \Rightarrow C_1)$. Suppose for all $\Lambda' \in U_c$, if $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$ then $[\psi]^+\chi \in \Lambda'$. Suppose $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \not\models [\psi]^+\chi$. Hence, there exists $T \in X_{\Gamma_0}$ such that $|\Gamma| \in T$, $T = \{|\Pi| \in S_\Box(\Delta) : \mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta)) \models \psi\}$ and $\mathcal{M}_{\Gamma_0}, (|\Gamma|, T) \not\models \chi$. Thus, $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models \psi$. Since $\Gamma \equiv \Theta$, $\Delta\ R_\Box\ \Theta$, $\psi \in P$ and for all $\Lambda' \in U_c$, if $\Gamma \equiv \Lambda'$ and $\Delta\ R_\Box\ \Lambda'$ then $[\psi]^+\chi \in \Lambda'$, therefore $\psi \in \Theta$ and $[\psi]^+\chi \in \Theta$. By Lemma 6.1, $\langle\psi\rangle^+\chi \in \Theta$. By Lemma 7.4, let $\Lambda \in U_c$ be such that $\Theta\ R_{[\psi]^+}\ \Lambda$ and $\chi \in \Lambda$. By Lemma 7.6, $[\psi]^+\Theta = \Lambda$. By Proposition 8.1, since $\Delta\ R_\Box\ \Theta$ and $\psi \in \Theta$, therefore $T = S_\Box(\Lambda)$. Since $\mathcal{M}_{\Gamma_0}, (|\Gamma|, T) \not\models \chi$, therefore $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Lambda)) \not\models \chi$. Since $\Gamma \equiv \Theta$ and $\Theta\ R_{[\psi]^+}\ \Lambda$, therefore $\Gamma \equiv \Lambda$. Since $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Lambda)) \not\models \chi$ and $\chi \in P$, therefore $\chi \notin \Lambda$: a contradiction. Consequently, $\mathcal{M}_{\Gamma_0}, (|\Gamma|, S_\Box(\Delta)) \models [\psi]^+\chi$.

Proof of Claim (a): (i) Let $\Pi' \in R^\psi_\Box(\Delta')$. Hence, $\Pi' \in R_\Box(\Delta')$ and $\psi \in \Pi'$. Since $\psi \in P$, therefore $\mathcal{M}_{\Gamma_0}, (|\Pi'|, S_\Box(\Delta') \models \psi$. Since $S_\Box(\Delta) = \{|\Pi| \in S_\Box(\Delta') : \mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta') \models \psi\}$, therefore $|\Pi'| \in S_\Box(\Delta)$. Let $\Pi \in U_c$ be such that $\Pi' \equiv \Pi$ and $\Delta\ R_\Box\ \Pi$. Thus, $\Pi \in R_\Box(\Delta)$ and $\Pi' \equiv \Pi$.
(ii) Let $\Pi \in R_\Box(\Delta)$. Since $S_\Box(\Delta) = \{|\Pi| \in S_\Box(\Delta') : \mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta') \models \psi\}$, therefore $|\Pi| \in S_\Box(\Delta')$ and $\mathcal{M}_{\Gamma_0}, (|\Pi|, S_\Box(\Delta') \models \psi$. Let $\Pi' \in U_c$ be such that $\Pi \equiv \Pi'$, $\Delta'\ R_\Box\ \Pi'$ and $\psi \in \Pi'$. Such $\Pi' \in U_c$ exists because $\psi \in P$.

Hence, $\Pi' \in R^{\psi}_{\Box}(\Delta')$ and $\Pi' \equiv \Pi$.

Proof of Claim (b)**:** Suppose $Q = Tri$ and $[\psi]^{+}\Theta' \not\subseteq \Theta$. Hence, there exists a formula φ such that $\varphi \in [\psi]^{+}\Theta'$ and $\varphi \notin \Theta$. Since $Q = Tri$, therefore $(deg(\varphi)+1, 0, \varphi) \in Q$. By condition (D_1), since $\Theta' \in R^{\psi}_{\Box}(\Delta')$, $\Theta \in R_{\Box}(\Delta)$ and $\Theta' \equiv \Theta$, therefore if $\varphi \in [\psi]^{+}\Theta'$ then $\varphi \in \Theta$. Since $\varphi \in [\psi]^{+}\Theta'$, therefore $\varphi \in \Theta$: a contradiction.

Proof of Claim (c)**:** The proof is done by induction on (d, m, φ), using the well-founded partial order $\ll$ on Tri defined as follows:

- $(d, m, \varphi) \ll (d', m', \varphi')$ iff one of the 3 following conditions holds:

(i) $d < d'$,
(ii) $d = d'$ and $m < m'$,
(iii) $d = d'$, $m = m'$ and $size(\varphi) < size(\varphi')$.

Let $(d, m, \varphi) \in Tri$ be such that for all $(d', m', \varphi') \in Tri$, if $(d', m', \varphi') \ll (d, m, \varphi)$ then $(d', m', \varphi') \in Q$. We demonstrate $(d, m, \varphi) \in Q$. Let $\varphi_1, \ldots, \varphi_m$ be formulas such that $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_m)) + deg(\varphi) \leq d$, $s_1, \ldots, s_m \in \{+, -\}$, $\Pi' \in R^{\psi}_{\Box}(\Delta')$ and $\Pi \in R_{\Box}(\Delta)$ be such that $\Pi' \equiv \Pi$. We demonstrate the 2 above conditions D_1 and D_2. Since $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_m)) + deg(\varphi) \leq d$, therefore $d \geq 4$. We consider the following 2 cases.
Case $m = 0$**.**
(D_1). Suppose $\varphi \in [\psi]^{+}\Pi'$. We demonstrate $\varphi \in \Pi$.
Subcase $\varphi = p$**.** Since $p \in [\psi]^{+}\Pi'$, therefore $[\psi]^{+}p \in \Pi'$. By Lemma 6.1, $\psi \to p \in \Pi'$. Since $\psi \in \Pi'$, therefore $p \in \Pi'$. Since $\Pi' \equiv \Pi$, therefore using (A_{14}) and Lemma 6.1, $p \in \Pi$.
Subcase $\varphi = \bot$**.** Since $\bot \in [\psi]^{+}\Pi'$, therefore $[\psi]^{+}\bot \in \Pi'$. Hence, using (A_{12}), $\psi \notin \Pi'$: a contradiction.
Subcase $\varphi = \neg\varphi'$**.** Since $\neg\varphi' \in [\psi]^{+}\Pi'$, therefore $[\psi]^{+}\neg\varphi' \in \Pi'$. By Lemma 6.1, $\psi \to \neg[\psi]^{+}\varphi' \in \Pi'$. Since $\psi \in \Pi'$, therefore $\neg[\psi]^{+}\varphi' \in \Pi'$. Hence, $[\psi]^{+}\varphi' \notin \Pi'$. Thus, $\varphi' \notin [\psi]^{+}\Pi'$. Obviously, $(d, 0, \varphi') \ll (d, 0, \neg\varphi')$. Consequently, $(d, 0, \varphi') \in Q$. Since $\varphi' \notin [\psi]^{+}\Pi'$, therefore $\varphi' \notin \Pi$. Hence, $\neg\varphi' \in \Pi$.
Subcase $\varphi = \varphi' \vee \varphi''$**.** Since $\varphi' \vee \varphi'' \in [\psi]^{+}\Pi'$, therefore $[\psi]^{+}(\varphi' \vee \varphi'') \in \Pi'$. By Lemma 6.1, $[\psi]^{+}\varphi' \in \Pi'$ or $[\psi]^{+}\varphi'' \in \Pi'$. Obviously, $(d, 0, \varphi') \ll (d, 0, \varphi' \vee \varphi'')$ and $(d, 0, \varphi'') \ll (d, 0, \varphi' \vee \varphi'')$. Consequently, $(d, 0, \varphi') \in Q$ and $(d, 0, \varphi'') \in Q$. Since $[\psi]^{+}\varphi' \in \Pi'$ or $[\psi]^{+}\varphi'' \in \Pi'$, therefore $\varphi' \in \Pi$ or $\varphi'' \in \Pi$. Hence, $\varphi' \vee \varphi'' \in \Pi$.
Subcase $\varphi = \Box\varphi'$**.** Since $\Box\varphi' \in [\psi]^{+}\Pi'$, therefore $[\psi]^{+}\Box\varphi' \in \Pi'$. Suppose $\Box\varphi' \notin \Pi$. By Lemma 7.2, let $\Pi_1 \in U_c$ be such that $\Pi\ R_{\Box}\ \Pi_1$ and $\varphi' \notin \Pi_1$. Since $\Pi \in R_{\Box}(\Delta)$, therefore $\Pi_1 \in R_{\Box}(\Delta)$. By item (ii) of Claim (a), let $\Pi'_1 \in R^{\psi}_{\Box}(\Delta')$ be such that $\Pi'_1 \equiv \Pi_1$. Obviously, $(d, 0, \varphi') \ll (d, 0, \Box\varphi')$. Consequently, $(d, 0, \varphi') \in Q$. Since $\varphi' \notin \Pi_1$, therefore $\varphi' \notin [\psi]^{+}\Pi'_1$. Hence, $[\psi]^{+}\varphi' \notin \Pi'_1$. Since $\Pi' \in R^{\psi}_{\Box}(\Delta')$ and $\Pi'_1 \in R^{\psi}_{\Box}(\Delta')$, therefore $\Box[\psi]^{+}\varphi' \notin \Pi'$. Thus, using (A_{16}), $\langle\psi\rangle^{+}\Box\varphi' \notin \Pi'$. Since $\psi \in \Pi'$, therefore

by Lemma 6.1, $[\psi]^+\Box\varphi' \notin \Pi'$: a contradiction.
Subcase $\varphi = [\varphi']^s\varphi''$. Since $[\varphi']^s\varphi'' \in [\psi]^+\Pi'$, therefore $[\psi]^+[\varphi']^s\varphi'' \in \Pi'$. Suppose $[\varphi']^s\varphi'' \notin \Pi$. By Lemma 7.4, let $\Pi_1 \in U_c$ be such that $\Pi\ R_{[\varphi']^s}\ \Pi_1$ and $\varphi'' \notin \Pi_1$. Hence, $\neg\varphi'' \in \Pi_1$. Obviously, $(d-1, 1, \neg\varphi'') \ll (d, 0, [\varphi']^s\varphi'')$. Consequently, $(d-1, 1, \neg\varphi'') \in Q$. Since $2 + deg([\varphi']^s\varphi'') \leq d$, therefore $deg(\varphi') + deg(\neg\varphi'') \leq d-1$. Since $(d-1, 1, \neg\varphi'') \in Q$, $\Pi\ R_{[\varphi']^s}\ \Pi_1$ and $\neg\varphi'' \in \Pi_1$, therefore let $\Pi'_1 \in U_c$ be such that $[\psi]^+\Pi'\ R_{[\varphi']^s}\ \Pi'_1$ and $\neg\varphi'' \in \Pi'_1$. Thus, $\langle\varphi'\rangle^s\neg\varphi'' \in [\psi]^+\Pi'$. Consequently, $[\psi]^+\langle\varphi'\rangle^s\neg\varphi'' \in \Pi'$. Since $[\psi]^+[\varphi']^s\varphi'' \in \Pi'$, therefore $[\psi]^+\bot \in \Pi'$. Hence, using (A_{12}), $\psi \notin \Pi'$. Thus, $\Pi' \notin R^{\psi}_{\Box}(\Delta')$: a contradiction.
(D_2). Suppose $\varphi \in \Pi$. We demonstrate $\varphi \in [\psi]^+\Pi'$.
Subcase $\varphi = p$. Since $p \in \Pi$ and $\Pi' \equiv \Pi$, therefore $[\psi]^+p \in \Pi'$. Hence, $p \in [\psi]^+\Pi'$.
Subcase $\varphi = \bot$. Obviously, $\bot \notin \Pi$.
Subcase $\varphi = \neg\varphi'$. Since $\neg\varphi' \in \Pi$, therefore $\varphi' \notin \Pi$. Since $(d, 0, \varphi') \ll (d, 0, \neg\varphi')$, therefore $(d, 0, \varphi') \in Q$. Since $\varphi' \notin \Pi$, therefore $\varphi' \notin [\psi]^+\Pi'$. Hence, $[\psi]^+\varphi' \notin \Pi'$. Thus, $\langle\psi\rangle^+\neg\varphi' \in \Pi'$. Consequently, using (A_{10}), $[\psi]^+\neg\varphi' \in \Pi'$.
Subcase $\varphi = \varphi' \vee \varphi''$. Since $\varphi' \vee \varphi'' \in \Pi$, therefore $\varphi' \in \Pi$ or $\varphi'' \in \Pi$. Without loss of generality, suppose $\varphi' \in \Pi$. Since $(d, 0, \varphi') \ll (d, 0, \varphi' \vee \varphi'')$, therefore $(d, 0, \varphi') \in Q$. Since $\varphi' \in \Pi$, therefore $\varphi' \in [\psi]^+\Pi'$. Hence, $[\psi]^+\varphi' \in \Pi'$. Thus, $[\psi]^+(\varphi' \vee \varphi'') \in \Pi'$. Consequently, $\varphi' \vee \varphi'' \in [\psi]^+\Pi'$.
Subcase $\varphi = \Box\varphi'$. Suppose $\Box\varphi' \notin [\psi]^+\Pi'$. Hence, $[\psi]^+\Box\varphi' \notin \Pi'$. Thus, using (A_{17}), $\Box[\psi]^+\varphi' \notin \Pi'$. By Lemma 7.2, let $\Pi'_1 \in U_c$ be such that $\Pi'\ R_{\Box}\ \Pi'_1$ and $[\psi]^+\varphi' \notin \Pi'_1$. Consequently, $\varphi' \notin [\psi]^+\Pi'_1$ and using (A_{11}), $\psi \in \Pi'_1$. Since $\Pi' \in R_{\Box}(\Delta')$ and $\Pi'\ R_{\Box}\ \Pi'_1$, therefore $\Pi'_1 \in R^{\psi}_{\Box}(\Delta')$. By item (i) of Claim (a), let $\Pi_1 \in R_{\Box}(\Delta)$ be such that $\Pi'_1 \equiv \Pi_1$. Obviously, $(d, 0, \varphi') \ll (d, 0, \Box\varphi')$. Consequently, $(d, 0, \varphi') \in Q$. Since $\varphi' \notin [\psi]^+\Pi'_1$, therefore $\varphi' \notin \Pi_1$. Since $\Pi \in R_{\Box}(\Delta)$ and $\Pi_1 \in R_{\Box}(\Delta)$, therefore $\Pi\ R_{\Box}\ \Pi_1$. Since $\varphi' \notin \Pi_1$, therefore $\Box\varphi' \notin \Pi$: a contradiction.
Subcase $\varphi = [\varphi']^s\varphi''$. Suppose $[\varphi']^s\varphi'' \notin [\psi]^+\Pi'$. By Lemma 7.4, let $\Pi'_1 \in U_c$ be such that $[\psi]^+\Pi'\ R_{[\varphi']^s}\ \Pi'_1$ and $\varphi'' \notin \Pi'_1$. Hence, $\neg\varphi'' \in \Pi'_1$. Obviously, $(d-1, 1, \neg\varphi'') \ll (d, 0, [\varphi']^s\varphi'')$. Consequently, $(d-1, 1, \neg\varphi'') \in Q$. Since $2 + deg([\varphi']^s\varphi'') \leq d$, therefore $deg(\varphi') + deg(\neg\varphi'') \leq d-1$. Since $(d-1, 1, \neg\varphi'') \in Q$, $[\psi]^+\Pi'\ R_{[\varphi']^s}\ \Pi'_1$ and $\neg\varphi'' \in \Pi'_1$, therefore let $\Pi_1 \in U_c$ be such that $\Pi\ R_{[\varphi']^s}\ \Pi_1$ and $\neg\varphi'' \in \Pi_1$. Thus, $[\varphi']^s\varphi'' \notin \Pi$: a contradiction.
Case $m \geq 1$.
(D_1). Let $\Pi'_1, \ldots, \Pi'_m \in U_c$ be such that $[\psi]^+\Pi'\ R_{[\varphi_1]^{s_1}}\ \Pi'_1$, $\Pi'_1\ R_{[\varphi_2]^{s_2}}\ \Pi'_2$, $\ldots$, $\Pi'_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi'_m$. If $\varphi \in \Pi'_m$ then let $\varphi' = \varphi$ else let $\varphi' = \neg\varphi$. Obviously, $deg(\varphi') = deg(\varphi)$ and $\langle\varphi_m\rangle^{s_m}\varphi' \in \Pi'_{m-1}$. Moreover, $(d, m-1, \langle\varphi_m\rangle^{s_m}\varphi') \ll (d, m, \varphi)$. Hence, $(d, m-1, \langle\varphi_m\rangle^{s_m}\varphi') \in Q$. Since $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_{m-1})) + deg(\langle\varphi_m\rangle^{s_m}\varphi') = (deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_{m-1})) + deg(\varphi_m) + deg(\varphi')$, $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_m)) + deg(\varphi) \leq d$ and $deg(\varphi') = deg(\varphi)$, therefore $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_{m-1})) + deg(\langle\varphi_m\rangle^{s_m}\varphi') \leq d$. Since $[\psi]^+\Pi'\ R_{[\varphi_1]^{s_1}}\ \Pi'_1$, $\Pi'_1\ R_{[\varphi_2]^{s_2}}\ \Pi'_2$, $\ldots$, $\Pi'_{m-2}\ R_{[\varphi_{m-1}]^{s_{m-1}}}\ \Pi'_{m-1}$, $\langle\varphi_m\rangle^{s_m}\varphi' \in$

Π'_{m-1} and $(d, m-1, \langle\varphi_m\rangle^{s_m}\varphi') \in Q$, therefore let $\Pi_1, \ldots, \Pi_{m-1} \in U_c$ be such that $\Pi\ R_{[\varphi_1]^{s_1}}\ \Pi_1$, $\Pi_1\ R_{[\varphi_2]^{s_2}}\ \Pi_2$, …, $\Pi_{m-2}\ R_{[\varphi_{m-1}]^{s_{m-1}}}\ \Pi_{m-1}$ and $\langle\varphi_m\rangle^{s_m}\varphi' \in \Pi_{m-1}$. Thus, let $\Pi_m \in U_c$ be such that $\Pi_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi_m$ and $\varphi' \in \Pi_m$. Consequently, $\Pi_1, \ldots, \Pi_m \in U_c$ are such that $\Pi\ R_{[\varphi_1]^{s_1}}\ \Pi_1$, $\Pi_1\ R_{[\varphi_2]^{s_2}}\ \Pi_2$, …, $\Pi_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi_m$ and if $\varphi \in \Pi'_m$ then $\varphi \in \Pi_m$.

(D_2). Let $\Pi_1, \ldots, \Pi_m \in U_c$ be such that $\Pi\ R_{[\varphi_1]^{s_1}}\ \Pi_1$, $\Pi_1\ R_{[\varphi_2]^{s_2}}\ \Pi_2$, …, $\Pi_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi_m$. If $\varphi \in \Pi_m$ then let $\varphi' = \varphi$ else let $\varphi' = \neg\varphi$. Obviously, $deg(\varphi') = deg(\varphi)$ and $\langle\varphi_m\rangle^{s_m}\varphi' \in \Pi_{m-1}$. Moreover, $(d, m-1, \langle\varphi_m\rangle^{s_m}\varphi') \ll (d, m, \varphi)$. Hence, $(d, m-1, \langle\varphi_m\rangle^{s_m}\varphi') \in Q$. Since $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_{m-1})) + deg(\langle\varphi_m\rangle^{s_m}\varphi') = (deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_{m-1})) + deg(\varphi_m) + deg(\varphi')$, $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_m)) + deg(\varphi) \leq d$ and $deg(\varphi') = deg(\varphi)$, therefore $(deg(\varphi_1) \cdot \ldots \cdot deg(\varphi_{m-1})) + deg(\langle\varphi_m\rangle^{s_m}\varphi') \leq d$. Since $\Pi\ R_{[\varphi_1]^{s_1}}\ \Pi_1$, $\Pi_1\ R_{[\varphi_2]^{s_2}}\ \Pi_2$, …, $\Pi_{m-2}\ R_{[\varphi_{m-1}]^{s_{m-1}}}\ \Pi_{m-1}$, $\langle\varphi_m\rangle^{s_m}\varphi' \in \Pi_{m-1}$ and $(d, m-1, \langle\varphi_m\rangle^{s_m}\varphi') \in Q$, therefore let $\Pi'_1, \ldots, \Pi'_{m-1} \in U_c$ be such that $[\psi]^+\Pi'\ R_{[\varphi_1]^{s_1}}\ \Pi'_1$, $\Pi'_1\ R_{[\varphi_2]^{s_2}}\ \Pi'_2$, …, $\Pi'_{m-2}\ R_{[\varphi_{m-1}]^{s_{m-1}}}\ \Pi'_{m-1}$ and $\langle\varphi_m\rangle^{s_m}\varphi' \in \Pi'_{m-1}$. Thus, let $\Pi'_m \in U_c$ be such that $\Pi'_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi'_m$ and $\varphi' \in \Pi'_m$. Consequently, $\Pi'_1, \ldots, \Pi'_m \in U_c$ are such that $[\psi]^+\Pi'\ R_{[\varphi_1]^{s_1}}\ \Pi'_1$, $\Pi'_1\ R_{[\varphi_2]^{s_2}}\ \Pi'_2$, …, $\Pi'_{m-1}\ R_{[\varphi_m]^{s_m}}\ \Pi'_m$ and if $\varphi \in \Pi_m$ then $\varphi \in \Pi'_m$.

Proof of Lemma 8.3: Let $\Gamma_0 \in U_c$. Using (A_{12}), by Proposition 8.2, for all formulas ϕ, $\mathcal{M}_{\Gamma_0} \models \phi \to \langle\phi\rangle^+\top$. By Lemma 3.1, $\mathcal{M}_{\Gamma_0}$ is free.