# A cut-free sequent calculus for the logic of subset spaces

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**Abstract**

Following the tradition of labelled sequent calculi for modal logics, we present a one-sided, cut-free sequent calculus for the bimodal logic of subset spaces. In labelled sequent calculi, semantical notions are internalised into the calculus, and we take care to choose them close to the original interpretation of the system. To achieve this, we introduce a variation of the standard method, considering structured labels instead of simple tokens, in our particular case pairs of labels. With this new device, we can formulate a calculus with extremely simple frame rules and good proof-theoretical properties. The logical rules are invertible, structural rules are admissible. We show the admissibility of cut and relate our system to the well-known Hilbert-style axiomatisation of the logic. Finally, we present a direct proof of completeness based on proof search.

*Keywords:* proof theory, cut-free sequent calculus, labelled deduction, direct completeness proof, logic of subset spaces.

## 1 Introduction

The logic of *subset spaces* SSL discussed here is a bimodal logic introduced in [1] for formalising reasoning about points and sets. Its extension *topologic* can be considered a refinement of Tarski's and McKinsey's topological interpretation [20,15] for the modal system **S4**. SSL is also called a logic of *knowledge* and *effort*. The relation to epistemic logic is investigated further in [17]. More recently, an interpretation of the language of public announcement logic in subset models was given [21]. Several extensions of the language of SSL have been studied, for example the addition of an *overlap operator* as a third modality [11] or announcement operators [2]. In the present work, however, we study the original language and its meaning given by subset spaces.

Subset frames consist of a set $X$ of *points* and a collection $\mathcal{O}$ of non-empty subsets of $X$ called *opens*. Worlds are pairs $(x, u)$ where $x$ is a point and $u$ is an open containing $x$. The first set $\mathrm{K}, \mathrm{L}$ of SSL-modalities corresponds to quantification over points in the same environment, while the second set $\Box, \Diamond$

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refers to the worlds obtained by shrinking the environment of a fixed point $x$. So the relation $\supseteq$ for opens determines the $\Box\Diamond$-reachability. A *sound* and *complete* Hilbert style axiomatisation is presented in [1]. It combines **S4**-axioms for $\Box, \Diamond$ with **S5**-axioms for $\mathrm{K}, \mathrm{L}$ and further axioms known as *persistence* for literals and *cross axioms*. As **S5** is contained as a subsystem, a corresponding *cut-free sequent calculus* is not straightforward (see [19] for a discussion of the case of **S5**), and the combination with a second set of modalities generates further difficulties.

*Labelled calculi* provide not only a solution for **S5** but also a general method to construct sequent systems for modal logics, see [18,7]. In that approach, the semantics is to a certain extent internalised into the calculus. The labels denote worlds in a Kripke frame. The basic judgements of the calculus have the form $x\colon A$ or $xRy$ which can be read as "$A$ holds at $x$" or "$y$ is reachable from $x$", respectively. In addition to the logical rules, one has *frame rules* that reflect the conditions for the Kripke frames of the logic.

We want to define a labelled calculus in that style based on subset frames. Corresponding to the structure of worlds in subset spaces, we use *pairs* $(x, u)$ of simple labels $x, u$ in judgements $(x, u) : A$ of our calculus and introduce formal judgements for "$(x, u)$ is a world" and "$u$ can be shrunk to obtain $v$".[2] The *frame rules* of the calculus reflect basic properties of these relations. From a semantic point of view, we generalise the class of models: the second components of pairs need not be sets and relations $\mathcal{W}$ and $\mathcal{R}$ are included in the frame, which have to satisfy some essential conditions but need not be identical to $\in$ and $\supseteq$. We call the elements of this more general class of models *abstract subset spaces*. As we keep the basic structure of pairs and the frame conditions for $\mathcal{W}$ and $\mathcal{R}$ are satisfied by $\in$ and $\supseteq$, subset spaces are a *special case* of abstract subset spaces, without any transformation of primitive notions.

Now the setting is different from the standard labelled systems but the general strategy can be employed to develop a cut-free calculus. In contrast to [18], we use a one-sided sequent system in the Schütte-Tait style. The logical rules correspond to the right rules of a two-sided system, the dual left rules are avoided. This cuts down the number of rules, although we retain all modalities. The interpretation of the modal operators as explained above and the conditions for abstract subset spaces determine the rules of the calculus.

As SSL does not have the finite model property w.r.t. subset spaces, the class of *cross axiom models* has also been introduced in [1] and has been used for the proof of the decidability. Alternatively to the approach presented here, we could have chosen these as the starting point and applied directly the method presented in [18], as all frame conditions satisfy the prerequisites. This leads to a cut-free system for SSL ([6]), as validity in all subset spaces and validity in all cross axiom models coincide but then the internalised semantics is significantly different from the original one. Obviously, the argument that is formalised in the deduction then uses the reachability relations instead of the notions given

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[2] More precisely for their negations, see below.

by the models itself. More important, the frame rules are no longer regular rule schemes. The accessibility relations in cross axiom frames must satisfy the so-called *cross condition*. This can be rewritten to a *geometric formula* in the sense of [18] but turning it into a proof rule in the natural way yields a *frame* rule which — read from bottom to top — generates new worlds via the involved eigenvariable.

In contrast to this, the requirements for abstract subset spaces are just closure conditions. Given any $\mathcal{W}_0 \subseteq X \times \mathcal{O}$ and $\mathcal{R}_0 \subseteq \mathcal{O} \times \mathcal{O}$, there is a least extension to an abstract subset space $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ and this can be obtained by a combination of standard relational operations (composition, inversion, reflexive-transitive closure). The corresponding frame rules are very simple. They are not subject to eigenvariable conditions. They could be readily replaced by a computation of this closure alternating with the logical rules or by complex application conditions for the logical rules that refer to that closure.

We proceed as follows: In Section 2 we present some basics concerning the logic of subset spaces and sequent calculi. Based on the model class of abstract subset spaces presented in Section 3, we develop a labelled calculus **LSSL-p** for SSL in Section 4. We show several proof-theoretic properties in Sections 5 and 6, in particular the invertibility of logical rules and the admissibility of weakening, contraction and cut. Derivations in **LSSL-p** of the Hilbert axioms from [1] are presented. Completeness, however, is proved directly in the style of [13,12]. In contrast to the proof in [1], the argument in Section 7 shows how to produce a derivation for valid formulas and yields a (in general infinite) countermodel for non-valid formulas.

## 2 Preliminaries

### 2.1 The logic of subset spaces

Following [1], a *subset frame* is a pair $\mathcal{X} = (X, \mathcal{O})$ where $X$ is a set of *points* and $\mathcal{O}$ is a set of non-empty subsets of $X$ called *opens*. We presuppose a fixed set PV of propositional letters. The *formulas* of the logic of subset spaces are built from the elements of PV using propositional connectives and the modalities $\Box, \Diamond, \mathrm{K}, \mathrm{L}$ where $\Box, \Diamond$ are dual to each other and so are $\mathrm{K}, \mathrm{L}$. The value of a propositional letter in a particular world is a truth value. For us, a valuation for a subset frame is a mapping $\mathcal{V} : X \to (\mathrm{PV} \to \mathbb{B})$ where $\mathbb{B}$ denotes the set of Boolean truth values, and a *subset space* $\mathcal{X} = (X, \mathcal{O}, \mathcal{V})$ consists of a subset frame $(X, \mathcal{O})$ and a valuation $\mathcal{V}$ for it. A *world* $(x, u)$ consists of a point $x \in X$ and an open $u$ that contains it. The *satisfaction relation* $\models_{\mathcal{X}}$ is given by the usual interpretation of the propositional connectives plus the following conditions for all $(x, u) \in X \times \mathcal{O}$ such that $x \in u$ and arbitrary formulas $A$:

$x, u \models_{\mathcal{X}} \mathrm{K}A$ iff $y, u \models_{\mathcal{X}} A$ for all $y \in u$
$x, u \models_{\mathcal{X}} \Box A$ iff $x, v \models_{\mathcal{X}} A$ for all $v \in \mathcal{O}$ such that $x \in v \subseteq u$
$x, u \models_{\mathcal{X}} \mathrm{L}A$ iff there exists $y \in u$ such that $y, u \models_{\mathcal{X}} A$
$x, u \models_{\mathcal{X}} \Diamond A$ iff there exists $v \in \mathcal{O}$ such that $x \in v \subseteq u$ and $x, v \models_{\mathcal{X}} A$

Here $x$ is a point and $u$ an open such that $x \in u$. Hence validity in a subset frame is just validity in the corresponding Kripke frame $(\mathcal{W}, \mathcal{S}, \mathcal{R})$ with the set $\mathcal{W} := \{(x, u) \in X \times \mathcal{O} \mid x \in u\}$ of worlds and the accessibility relations

$$\begin{aligned} \mathcal{S} &:= \{((x, u), (y, u)) \mid x, y \in X \text{ and } \{x, y\} \subseteq u \in \mathcal{O}\} \\ \mathcal{R} &:= \{((x, u), (x, v)) \mid x \in v \subseteq u\} \end{aligned}$$

for $\Box, \Diamond$ and $\textsc{k}, \textsc{l}$ respectively.

Cross axiom frames and cross axiom models are introduced in [1] in order to prove the decidability of subset space logic (also see [14] for a simplified proof). A *cross axiom frame* $(\mathcal{W}, \mathcal{S}, \mathcal{R})$ consists of a set $\mathcal{W}$, an equivalence relation $\mathcal{S}$ on $\mathcal{W}$ and a preorder $\mathcal{R}$ on $\mathcal{W}$ so that $\mathcal{R};\mathcal{S} \subseteq \mathcal{S};\mathcal{R}$. Here and in the sequel we use standard notation for operations on relations: ";" stands for relational composition (not $\circ$), $\cdot^+$ for the transitive closure and $\cdot^*$ for the reflexive-transitive closure. A *cross axiom model* $(\mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{V})$ is a cross axiom frame $(\mathcal{W}, \mathcal{S}, \mathcal{R})$ together with a valuation $\mathcal{V} : \mathcal{W} \to (\mathrm{PV} \to \mathbb{B})$ so that $\mathcal{V}(w) = \mathcal{V}(w')$ whenever $(w, w') \in \mathcal{R}$. It can easily be checked that the transformation for subset frames into Kripke models described above yields a cross axiom frame. Extending this with the valuation $\mathcal{V}' : \mathcal{W} \to (\mathrm{PV} \to \mathbb{B})$ given by $\mathcal{V}'((x, u)) := \mathcal{V}(x)$, we obtain a cross axiom model, in which the same formulas are valid. [3]

The reason for introducing cross axiom models as an auxiliary concept lies in the fact that they enjoy the finite model property, in contrast to subset spaces. In particular, this tells us that there are cross axiom models which are not isomorphic to a Kripke frame induced by a subset space. A characterisation of those cross axiom frames that are isomorphic copies of transformed subset frames is presented by Heinemann in [10]. However, validity in all subset models and validity in all cross axiom models coincide, similar for satisfiability. This is a consequence of the fact that the Hilbert-system given below is sound w.r.t. cross axiom models (and hence also w.r.t. subset spaces) and complete w.r.t. subset spaces (and hence also w.r.t. cross axiom models).

The axioms of *subset space logic*, see Table 1, are given in [1]. The instances of the axiom scheme (ca) are called *cross axioms*, and (pers) is the *persistence* for literals. Furthermore, we have **S5**-axioms for $\textsc{k}$ and **S4**-axioms for $\Box$. The rules of inference are *modus ponens* and the usual rules of *necessitation* for $\Box, \textsc{k}$. We denote the (Hilbert style) deductive system given by the axioms and rules in Table 1 by **HSS**.

In [1], the modalities $\Diamond, \textsc{l}$ (as well as $\vee, \to$) are defined notions. There, $\Diamond A$ stands for $\neg\Box\neg A$, and $\textsc{l}A$ stands for $\neg\textsc{k}\neg A$. As the focus is on the modalities, we prefer to keep all four of them as primitives, and reduce the number of logical operators in a different way: negation on non-atoms and implication are taken as defined. That means that $\wedge, \vee$ can be used freely in building a formula, while '$\to$' is excluded and '$\neg$' restricted to the case of propositional variables. This can also be understood as presupposing a second set of *negative*

[3] For the more general case of abstract subset spaces, see 3.3.

*Axioms* in the system **HSS**:
all substitution instances of tautologies of propositional logic
$(P \to \Box P) \wedge (\neg P \to \Box \neg P)$ (pers) for propositional letters $P$
$\textsc{k}\Box A \to \Box \textsc{k} A$ (ca)
$\textsc{k}A \to (A \wedge \textsc{kk}A)$ $\Box A \to (A \wedge \Box\Box A)$
$\textsc{k}(A \to B) \to (\textsc{k}A \to \textsc{k}B)$ $\Box(A \to B) \to (\Box A \to \Box B)$ $\textsc{l}A \to \textsc{kl}A$

*Rules*:

$$\frac{A \to B \qquad A}{B} \qquad \frac{A}{\Box A} \qquad \frac{A}{\textsc{k}A}$$

Table 1
The system **HSS** - axioms and rules of the logic of subset spaces

*literals* $\neg P, \neg Q, \neg P', \neg Q, \ldots$ equipped with a bijection '$\neg$', mapping *positive literals* (i.e. propositional variables) to negative literals. This mapping is used in the semantics of the language as well as in the logical axioms of the calculi. Negation *for non-atoms* is given by

$$\begin{array}{llll} \neg\Box A :\equiv \Diamond\neg A & \neg\Diamond A :\equiv \Box\neg A & \neg\textsc{k}A :\equiv \textsc{l}\neg A & \neg\textsc{l}A :\equiv \textsc{k}\neg A \\ \neg\neg P :\equiv P & \neg(A \wedge B) :\equiv \neg A \vee \neg B & \neg(A \vee B) :\equiv \neg A \wedge \neg B & \end{array}$$

and $A \to B$ stands for $\neg A \vee B$. For one-sided sequent systems, a significant simplification is achieved by using this defined negation for compound formulas. It is a prerequisite for the GS-calculi in [9] and part of the Schütte-Tait-style, which we will adopt for the calculus in Section 4.

## 2.2 One-sided labelled sequent calculi

As our starting point, we choose a propositional, one-sided sequent calculus in the Schütte-Tait style where weakening and contraction are absorbed into the logical rules, i.e. the propositional, cut-free part of the calculus **GS3** in [9]:

$$(\mathrm{ax})\,\frac{}{\Gamma, P, \neg P} \qquad (\wedge)\,\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad (\vee)\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

Here and in the sequel, sequents are multisets of formulas. A two-sided sequent $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ corresponds to $\neg A_1, \ldots, \neg A_m, B_1, \ldots, B_n$. Negri's system **G3K** [18] can readily be rewritten in the one-sided style. Originally, the elements of sequents are *relational atoms* $xRy$ or *labelled formulas* $x\colon A$, where $x, y$ are *labels* taken from a fixed set, $A$ is a modal formula, and $R$ is a binary relation symbol that stands for the accessibility relation. Logical axioms for the relational atoms are present but it is pointed out in [18] that they are only needed for deriving properties of the accessibility relation. Hence they can safely be removed. As a consequence, atoms $tRs$ would be needed in the original setting on the left side of sequents only. Correspondingly, they would occur *negated* only in the one-sided system. So we can introduce relational symbols $\overline{R}$ for the complement relation right from the beginning and avoid negation. Now we obtain the system **GS3K** in Table 2. Here $!(y)$ abbreviates

$$(\text{ax})\ \frac{}{\Gamma, x\colon P, x\colon \neg P} \qquad (\wedge)\ \frac{\Gamma, x\colon A \qquad \Gamma, x\colon B}{\Gamma, x\colon A \wedge B} \qquad (\vee)\ \frac{\Gamma, x\colon A, x\colon B}{\Gamma, x\colon A \vee B}$$

$$(\Box)\ \frac{\Gamma, x\,\overline{R}\,y, y\colon A}{\Gamma, x\colon \Box A}\,!(\text{y}) \qquad (\Diamond)\ \frac{\Gamma, x\,\overline{R}\,y, x\colon \Diamond A, y\colon A}{\Gamma, x\,\overline{R}\,y, x\colon \Diamond A}$$

Table 2
The system GS3K

the usual eigenvariable condition that $y$ does not occur in the conclusion.

In [18] a general method for generating cut-free sequent calculi for modal logics is presented. It applies to normal modal logics which are characterised by universal axioms or, more generally, geometric implications as frame conditions. The latter are formulas of the form $\forall \bar{x}\,(A \to B)$ where $A, B$ are formulas not containing[4] $\to$ or $\forall$. Geometric frame conditions can be transformed schematically into left rules of the calculus. For example, the conditions for **S4** and **S5** are universal formulas. Using $R$ as a symbol for the accessibility relation, they can be written as

| | | |
|---|---|---|
| (reflexivity) | $\forall x\,(xRx)$ | $(\mathbf{S4}, \mathbf{S5})$ |
| (transitivity) | $\forall x \forall y \forall z\,(xRy \wedge yRz \to xRz)$ | $(\mathbf{S4}, \mathbf{S5})$ |
| (symmetry) | $\forall x \forall y\,(xRy \to yRx)$ | $(\mathbf{S5})$ |

which yield the rules:

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\text{Ref} \qquad \frac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\,\text{Trans}$$

$$\frac{yRx, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\,\text{Sym}$$

By reformulating these for one-sided sequents we obtain

$$\frac{\Gamma, x\,\overline{R}\,x}{\Gamma}\,\text{Ref} \qquad \frac{\Gamma, x\,\overline{R}\,z, x\,\overline{R}\,y, y\,\overline{R}\,z}{\Gamma, x\,\overline{R}\,y, y\,\overline{R}\,z}\,\text{Trans} \qquad \frac{\Gamma, y\,\overline{R}\,x, x\,\overline{R}\,y}{\Gamma, x\,\overline{R}\,y}\,\text{Sym}$$

Adding the corresponding rules to **GS3K**, we obtain labelled systems for **S4** and **S5**. The two-sided versions of these systems, the general method, as well as systems for further modal logics are studied in detail in [18].

## 3 Abstract subset spaces

We introduce a class of models which is slightly more general than subset spaces. As in subset spaces, the worlds are *pairs* from a set $X \times \mathcal{O}$ but $\mathcal{O}$ need not consist of subsets of $X$. The relation $\mathcal{W}$ determines which pairs are indeed worlds. In subset spaces this is fixed to be $\in$. Similar to subset spaces, the accessibility

[4] In this context, negation is defined using $\to, \bot$, hence also excluded from $A, B$.

relation for K can be described as “equality of the second components” and the accessibility relation for $\Box$ is determined by a relation on $\mathcal{O}$ but the latter relation need no longer be $\supseteq$.

**Definition 3.1** An *abstract subset frame* $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ consists of sets $X, \mathcal{O}$, a relation $\mathcal{W} \subseteq X \times \mathcal{O}$ and a preorder $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{O}$ so that $\mathcal{W}; \mathcal{R}^{-1} \subseteq \mathcal{W}$. An *abstract subset space* $(X, \mathcal{O}, \mathcal{W}, \mathcal{R}, \mathcal{V})$ consists of an abstract subset frame $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ and a valuation $\mathcal{V} : X \to (\mathrm{PV} \to \mathbb{B})$.

Setting $\mathcal{W} = \{(x, u) \in X \times \mathcal{O} \mid x \in u\}$ and $\mathcal{R} = \{(u, v) \in \mathcal{O} \times \mathcal{O} \mid v \subseteq u\}$ turns every subset space into an abstract subset space. Choosing a set of properties for the definition of abstract subset frames can be interpreted as looking for a set of (simple and natural) axioms for $\in, \supseteq$ that is sufficient for our purpose. We use reflexivity and transitivity of $\supseteq$ as well as the obvious $\forall x \in X \forall u, v \in \mathcal{O}\, (x \in u \wedge u \subseteq v \to x \in v)$. Antisymmetry of $\supseteq$ is simply not needed.

The assignment of cross axiom models to subset spaces is generalised to the case of abstract subset spaces in the straightforward way:

**Definition 3.2** Let $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ be an abstract subset frame. The corresponding accessibility relations $\hat{\mathcal{S}}, \hat{\mathcal{R}}$ are defined as follows:

$$\begin{aligned} \hat{\mathcal{S}} &:= \{((x, u), (y, u)) \mid (x, u), (y, u) \in \mathcal{W}\} \\ \hat{\mathcal{R}} &:= \{((x, u), (x, v)) \mid (x, u), (x, v) \in \mathcal{W} \text{ and } (u, v) \in \mathcal{R}\} \end{aligned}$$

If $\mathcal{V} : X \to (\mathrm{PV} \to \mathbb{B})$ is a valuation for that frame, then the mapping $\hat{\mathcal{V}} : \mathcal{W} \to (\mathrm{PV} \to \mathbb{B})$ is given by $\hat{\mathcal{V}}(x, u) := \mathcal{V}(x)$ for all $(x, u) \in \mathcal{W}$.

**Lemma 3.3** *Let $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ be an abstract subset frame. Then $(\mathcal{W}, \hat{\mathcal{S}}, \hat{\mathcal{R}})$ is a cross axiom frame. If furthermore $\mathcal{V} : X \to (PV \to \mathbb{B})$ is a valuation for that abstract subset frame, then $(\mathcal{W}, \hat{\mathcal{S}}, \hat{\mathcal{R}}, \hat{\mathcal{V}})$ is a cross axiom model.*

**Proof.** Straightforward verification. We present the proof of the properties which are most characteristic for cross axiom models.
*Cross property*: Let $(x, u)\hat{\mathcal{R}}(y, v)$ and $(y, v)\hat{\mathcal{S}}(z, w)$. Then $x = y$, $v = w$, $(u, v) \in \mathcal{R}$, and the pairs $(x, u), (y, v), (z, w)$ are worlds in $\mathcal{W}$. As $\mathcal{W}; \mathcal{R}^{-1} \subseteq \mathcal{W}$, we can infer that $(z, u) \in \mathcal{W}$. Hence $(x, u)\hat{\mathcal{S}}(z, u)$ and $(z, u)\hat{\mathcal{R}}(z, v) = (z, w)$.
*Persistence*: Let $(x, u), (y, v) \in \hat{\mathcal{R}}$. Then $x = y$, and we have $\hat{\mathcal{V}}(x, u) = \mathcal{V}(x) = \mathcal{V}(y) = \hat{\mathcal{V}}(y, v)$. $\Box$

The validity of formulas in abstract subset spaces is defined as usual, using the accessibility relation $\hat{\mathcal{S}}$ for K, L and $\hat{\mathcal{R}}$ for $\Box, \Diamond$. Hence the use of K, L amounts to quantification over worlds with the same second component, and $\Box, \Diamond$ refer to all worlds with the same first and an $\mathcal{R}$-reachable second component. Furthermore, validity in an abstract subset space coincides with validity in the induced cross axiom model. The soundness of **HSS** w.r.t. abstract subset spaces is immediate from the soundness w.r.t. cross axiom models, and completeness follows from completeness w.r.t. subset spaces. So the difference lies in the class of models, not in the set of valid sentences.

Comparing abstract subset spaces with subset spaces, we find first that they are *conceptually* close. Some specific choices, however, are replaced by postulated properties, and this is part of the development of the rule set in Section 4. Second, we observe that cross axiom models induced by abstract subset spaces satisfy Heinemann's conditions in [10] for isomorphic copies of cross axiom models induced by subset spaces.

In contrast to subset spaces and cross axiom models, the requirements for abstract subset spaces are simple closure conditions. Hence, given arbitrary sets $\mathcal{W}_0 \subseteq X \times \mathcal{O}$ and $\mathcal{R}_0 \subseteq \mathcal{O} \times \mathcal{O}$, there is a unique least extension that is an abstract subset frame:

**Lemma 3.4** *Let $X, \mathcal{O}$ be sets and $\mathcal{W}_0, \mathcal{R}_0$ relations so that $\mathcal{W}_0 \subseteq X \times \mathcal{O}$ and $\mathcal{R}_0 \subseteq \mathcal{O} \times \mathcal{O}$. Then $(X, \mathcal{O}, (\mathcal{W}_0; (\mathcal{R}_0^*)^{-1}), \mathcal{R}_0^*)$ is the least abstract subset frame $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ so that $\mathcal{W}_0 \subseteq \mathcal{W}$ and $\mathcal{R}_0 \subseteq \mathcal{R}$. It is called the abstract subset frame* generated by $(X, \mathcal{O}, \mathcal{W}_0, \mathcal{R}_0)$.

**Proof.** The relation $\mathcal{R}_0^*$ is a preorder. Furthermore:

$$(\mathcal{W}_0; (\mathcal{R}_0^*)^{-1}); (\mathcal{R}_0^*)^{-1} = \mathcal{W}_0; (\mathcal{R}_0^*; \mathcal{R}_0^*)^{-1} \subseteq \mathcal{W}_0; (\mathcal{R}_0^*)^{-1}$$

Let $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ be an abstract subset frame satisfying $\mathcal{W}_0 \subseteq \mathcal{W}$ and $\mathcal{R}_0 \subseteq \mathcal{R}$. Then the reflexive-transitive closure $\mathcal{R}_0^*$ is a subset of $\mathcal{R}$, and consequently $\mathcal{W}_0; (\mathcal{R}_0^*)^{-1} \subseteq \mathcal{W}; \mathcal{R}^{-1} \subseteq \mathcal{W}$. □

# 4 The labelled sequent calculus LSSL-p

## 4.1 Axioms and rules of LSSL-p

Now we define a calculus **LSSL-p**, a **l**abelled calculus for **s**ubset **s**pace **l**ogic, following the general method of constructing labelled calculi but introducing **p**airs as labels.

For **LSSL-p**, we need two disjoint sets $L_1, L_2$ of labels. We use the symbols $x, y, z, x', x_1, \ldots$ for the elements of $L_1$ and $u, v, w, u', u_1, \ldots$ for the elements of $L_2$. Our judgements are relational atoms $x\,\overline{W}\,u$ or $u\,\overline{R}\,v$ or of the form $(x, u)\colon A$ where $(x, u) \in L_1 \times L_2$ and $A$ is an SSL-formula as given above. Some of the rules are subject to a condition which is abbreviated to $j(\ldots)$ and will be discussed below. The $!(\ldots)$ stands for the usual *eigenvariable* condition that the label does not occur in the conclusion.

The letters $\overline{W}$, $\overline{R}$ in formulas stand for the *complement* of the corresponding relations. The judgement $(x, u)\colon A$ should be read as "*if* $(x, u)$ is a world, then $A$ holds at $(x, u)$". Note that $(x, u)$ might be no world, in which case "$A$ holds at $(x, u)$" makes no sense. This is true for abstract subset spaces — where the set of worlds is given by $\mathcal{W}$ — as well as for the original subset spaces where $x, u \models \ldots$ is defined only in case that $x \in u$. To put it differently, the "term" $(x, u)$ is a partial term, as it may have no value in the given model. The statement *not* $x\,\overline{W}\,u$ — $(x, u)$ is a world — then corresponds to "$(x, u)$ denotes". From $(y, u)\colon A$ we could only deduce "if $(y, u)$ is a world then $(x,u)\colon \textsc{l}A$". [5] Instead of introducing $y\,\overline{W}\,u$ together with $(x,u)\colon \textsc{l}A$, we postulate that it is already present in the context. As $(y,u)\colon B$ corresponds to "if $(y,u)$ is a world then …", any judgement $(y,u)\colon B$ in the context would do the job. Consequently, in the formulation of the calculus we use the condition:

***j(y, u)* :** "The conclusion contains some **j**udgement $(y,u)\colon B$ or $y\,\overline{W}\,u$."

The rule (L) could be split into

$$\frac{\Gamma, y\,\overline{W}\,u, (x,u) : \textsc{l}A, (y,u) : A}{\Gamma, y\,\overline{W}\,u, (x,u) : \textsc{l}A} \qquad \frac{\Gamma, (y,u) : B, (x,u) : \textsc{l}A, (y,u) : A}{\Gamma, (y,u) : B, (x,u) : \textsc{l}A}$$

which makes the similarity to the usual rule for 'possibly $A$' more explicit but the condition above abbreviated by $j(y,u)$ provides a way to combine these two possibilities. As we interpret the judgements in subset spaces where only certain pairs are worlds, it seems adequate to read the complex label $(x,u)$ as a partial term but it comes with a straightforward totalisation: extend the domain to $X \times \mathcal{O}$ and map $(x,u)$ to the corresponding pair, then read $(x,u) : A$ (again) as: "$(x,u)$ is no world or it is a world, at which $A$ holds." This extension eliminates the partiality if desired. Still, "$(y,u)$ is no world" does not imply $(x,u) : A$. So the side condition will be kept in the calculus. Also note that in a proof search this side condition restricts the instantiation to worlds that are already present in the lower sequent. A similar remark applies to the $\Diamond$-rule, which is also subject to a condition $j(\ldots)$. Due to the $(\Diamond_{ref})$-rule, we do not need a reflexivity rule for $R$. The reflexivity of the accessibility relation for K/L is built into the system.

Now let us consider the necessitation rules. For soundness, the atom $x\,\overline{W}\,u$ in the premiss of the $\Box$ and K-rules would not be necessary. If preferred, we can generalise the rule so that $x\,\overline{W}\,u$ need not be present in the upper sequent. As weakening is admissible, this makes no big difference (except shortening some sequents in the derivation). We will, however, use the fact that an atom $x\,\overline{W}\,u$ can be contracted into the $(x,u)\colon \Box A$ or $(x,u)\colon \textsc{k}A$ built in a $(\Box)/(\textsc{k})$-inference. This improves the permutability of rules. To see this, consider a derivation ending with

$$(\Box)\,\frac{\Gamma\,,\, w\,\overline{R}\,u\,,\,(x,w)\colon \Diamond B\,,\,(x,u)\colon B\,,\,(x,v)\colon A\,,\,u\,\overline{R}\,v\,,x\,\overline{W}\,u}{(\Diamond)\,\dfrac{\Gamma\,,\, w\,\overline{R}\,u\,,\,(x,w)\colon \Diamond B\,,\,(x,u)\colon B\,,\,(x,u)\colon \Box A}{\Gamma\,,\, w\,\overline{R}\,u\,,\,(x,w)\colon \Diamond B\,,\,\boldsymbol{(x,u)}\colon \Box A}}$$

in which $(\Box)$ can be permuted downward.

The rules ($R$-trans) and ($RW$) just reflect the conditions for abstract subset frames. An example for the use of ($RW$) can be found in the derivation of the cross axiom in the proof of Lemma 5.5. Persistence can be combined

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[5] Compare this with Beeson's axiom ([3], p. 98)

$$A\{t/x\} \wedge t\downarrow \to \exists x A$$

where $t\downarrow$ stands for "$t$ denotes".

$$(\text{ax})\ \frac{}{\Gamma, (x,u)\colon P, (x,v)\colon \neg P} \qquad (\Diamond_{ref})\ \frac{\Gamma, (x,u)\colon \Diamond A, (x,u)\colon A}{\Gamma, (x,u)\colon \Diamond A}$$

$$(\Box)\ \frac{\Gamma, x\,\overline{W}\,u, u\,\overline{R}\,v, (x,v)\colon A}{\Gamma, (x,u)\colon \Box A}\ !(v) \qquad (\Diamond)\ \frac{\Gamma, u\,\overline{R}\,v, (x,u)\colon \Diamond A, (x,v)\colon A}{\Gamma, u\,\overline{R}\,v, (x,u)\colon \Diamond A}\ j(x,v)$$

$$(\text{K})\ \frac{\Gamma, x\,\overline{W}\,u, (y,u)\colon A}{\Gamma, (x,u)\colon \text{K}A}\ !(y) \qquad (\text{L})\ \frac{\Gamma, (x,u)\colon \text{L}A, (y,u)\colon A}{\Gamma, (x,u)\colon \text{L}A}\ j(y,u)$$

$$(\wedge)\ \frac{\Gamma, (x,u)\colon A \qquad \Gamma, (x,u)\colon B}{\Gamma, (x,u)\colon A \wedge B} \qquad (\vee)\ \frac{\Gamma, (x,u)\colon A, (x,u)\colon B}{\Gamma, (x,u)\colon A \vee B}$$

$$(R\text{-trans})\ \frac{\Gamma, u\,\overline{R}\,v, v\,\overline{R}\,w, u\,\overline{R}\,w}{\Gamma, u\,\overline{R}\,v, v\,\overline{R}\,w} \qquad (RW)\ \frac{\Gamma, v\,\overline{R}\,u, x\,\overline{W}\,v}{\Gamma, v\,\overline{R}\,u}\ j(x,u)$$

Table 3
System **LSSL-p**

conveniently with the logical axioms. Note that this axiom is simpler than the persistence condition for cross axiom models which refers to the $R$-accessibility relation. The full system is given in Table 3. We use $\vdash$ for derivability and $\vdash^n$ for the existence of a derivation of height $\leq n$.

**Definition 4.1** Let $\mathcal{M} = (X, \mathcal{O}, \mathcal{W}, \mathcal{R}, \mathcal{V})$ be an abstract subset spaces, and $\ell_1 : L_1 \to X$ and $\ell_2 : L_2 \to \mathcal{O}$ mappings. Then, based on the validity of formulas, we define the validity of judgements and sequents as follows:

$$\begin{array}{lll}(\mathcal{M}, \ell_1, \ell_2) \models (x,u) : A & \iff & ((\ell_1(x), \ell_2(u)) \in \mathcal{W} \text{ implies } (\ell_1(x), \ell_2(u)) \models_{\mathcal{M}} A) \\ (\mathcal{M}, \ell_1, \ell_2) \models x\,\overline{W}\,u & \iff & (\ell_1(x), \ell_2(u)) \notin \mathcal{W} \\ (\mathcal{M}, \ell_1, \ell_2) \models u\,\overline{R}\,v & \iff & (\ell_2(u), \ell_2(v)) \notin \mathcal{R} \\ (\mathcal{M}, \ell_1, \ell_2) \models \Gamma & \iff & (\mathcal{M}, \ell_1, \ell_2) \models J \text{ for some judgement } J \text{ in } \Gamma\end{array}$$

**Lemma 4.2 (Soundness)** *If* $\vdash \Gamma$ *then* $(\mathcal{M}, \ell_1, \ell_2) \models \Gamma$ *for all abstract subset spaces* $\mathcal{M} = (X, \mathcal{O}, \mathcal{W}, \mathcal{R}, \mathcal{V})$ *and mappings* $\ell_1 : L_1 \to X$ *and* $\ell_2 : L_2 \to \mathcal{O}$.

**Proof.** Induction on the height of a **LSSL-p** derivation. □

**Corollary 4.3** ***LSSL-p*** *is sound with respect to subset spaces.*

### 4.2 The role of pairs

We have presented the axioms and rules of **LSSL-p** and argued that it is rather natural, as it is close to the subset space semantics. Before we demonstrate that the calculus enjoys the desired proof-theoretical properties, we want to discuss its design in relation to alternative approaches.

Labelled sequent systems have been used before in connection with non-relational semantics. Gilbert and Maffezioli [8] for example develop sequent calculi for several modal logics which are weaker than the smallest normal modal logic. Semantics for these languages is usually based on *neighbourhood* frames. The calculi utilise a translation into a multi-modal system with normal modalities. Negri and Olivetti [16] present a sequent calculus for preferential conditional logic PCL. They internalise the weak neighbourhood semantics for PCL. Similar to our approach, the set of judgements is extended in both cases. In [8] we have relational judgements that do not state accessibility in the original setting but refer to accessibility in the multi-modal system. In [16] the extension is taken even further. In order to deal with the quantifier alternation in the semantical explanation of the conditional, new primitives for certain subexpressions of that definition are introduced. Note that the shift from the topological semantics (see [4]) to the bimodal system of topologic (see [1]), also eliminates the alternation of quantifiers in the semantic definition

$$\exists u \in \mathcal{O}(x \in u \wedge \forall y \in u \text{“}A \text{ holds at } y\text{”})$$

of “necessarily $A$”. The universal modality is mapped to $\Diamond\textsc{k}$ in topologic (see [1], p. 103). However, the worlds in topologic and in the weaker subset space logic contain a component ‘point’ as well as a component ‘set’. Points and sets play a role also in [8] and [16]. In [8] the labels in the calculus stand for worlds in the translated system, which are points or sets of points where the points are distinguished with the help of the modality ‘$\sigma$’. In [16], two types of labels are used instead. In contrast to both settings, the worlds in subset spaces are *pairs* of points and sets, and so we use pairs also in the judgements.

A noticeable feature of **LSSL-p** is the simplicity of the frame rules. They are based on universal axioms only. We could even replace them by complex application conditions for $\Diamond$ and $\textsc{l}$ that refer to the computed closure. To this end, let

$$\begin{aligned}
\mathbf{L}_1(\Gamma) &= \{x \in L_1 \mid x \text{ occurs in } \Gamma\} \\
\mathbf{L}_2(\Gamma) &= \{u \in L_2 \mid u \text{ occurs in } \Gamma\} \\
\mathbf{R}_0(\Gamma) &= \{(u,v) \mid u\,\overline{R}\,v \text{ occurs in } \Gamma\} \subseteq \mathbf{L}_2(\Gamma) \times \mathbf{L}_2(\Gamma) \\
\mathbf{W}_0(\Gamma) &= \{(x,u) \mid x\,\overline{W}\,u \text{ or some } (x,u)\colon A \text{ occurs in } \Gamma\} \\
\mathbf{R}(\Gamma) &= \mathbf{R}_0(\Gamma)^* \subseteq \mathbf{L}_2(\Gamma) \times \mathbf{L}_2(\Gamma) \\
\mathbf{W}(\Gamma) &= \mathbf{W}_0(\Gamma); \mathbf{R}(\Gamma)^{-1}
\end{aligned}$$

for multisets $\Gamma$ of judgements. If we generalise ($\Diamond$) and ($\textsc{l}$) to

$$(\Diamond^*)\ \frac{\Gamma, (x,u) : \Diamond A, (x,v) : A}{\Gamma, (x,u) : \Diamond A} \quad \begin{array}{ll} \text{if} & (x,v) \in \mathbf{W}(\Gamma, (x,u) : \Diamond A) \\ \text{and} & (u,v) \in \mathbf{R}(\Gamma, (x,u) : \Diamond A) \end{array}$$

$$(\textsc{l}^*)\ \frac{\Gamma, (x,u) : \textsc{l}A, (y,u) : A}{\Gamma, (x,u) : \textsc{l}A} \text{ if } (y,u) \in \mathbf{W}(\Gamma, (x,u) : \Diamond A)$$

then we can remove the frame rules. This is also the first step in the development of the ‘compressed’ version of the system which is used for proof search in Section 6.

Still, the cross axiom models offer an alternative way to define a calculus without introducing pairs [6]. The only frame condition that has not already been studied is the cross condition $\mathcal{R}; \mathcal{S} \subseteq \mathcal{S}; \mathcal{R}$ which can be transformed to the geometric formula $\forall x, y, y(xRy \wedge ySz \rightarrow \exists y'(xSy' \wedge y'Rz))$ and yields the cross rule:

$$\frac{\Gamma, x\,\overline{R}\,y, y\,\overline{S}\,z, x\,\overline{S}\,y', y'\,\overline{R}\,z}{\Gamma, x\,\overline{R}\,y, y\,\overline{S}\,z}$$

The set of derivable formulas of these two systems coincide but neither can be translated just by replacing every step locally by a sequence of steps of the other system. Passing from the system based on cross axiom models to **LSSL-p** would require choosing an appropriate substitution of pairs for simple labels so that the $R/S$-relations are reduced to the special form in abstract subset spaces and, in particular, the applications of the cross rule with their eigenvariables can be removed. The other direction is a bit simpler, as we can leave the labelled formulas unchanged. The frame judgements have to be translated according to the transformation of abstract subset spaces into cross axiom models. Instead of translating the applications of frame rules one by one, the block of frame rules needed for the computation of the closure should be transformed as a whole. Further analysis of the class of derivations obtained by this translation might be useful for advanced proof-theoretic investigations of SSL but then one could use **LSSL-p** right away.

## 5 Basic properties of LSSL-p

We start with some simple properties of the calculus.

**Lemma 5.1** *The following holds for **LSSL-p**:*

(i) (renaming) *Let $d$ be a derivation with endsequent $\Gamma$ and $x, y \in L_1$ (or $u, v \in L_2$) where $y$ (or $v$) does not occur in $d$. Then replacing every occurrence of $x$ in $d$ by $y$ (or $u$ by $v$ respectively) yields a derivation of the endsequent $\Gamma\{y/x\}$ (or $\Gamma\{v/u\}$ respectively).*

(ii) (label substitution) *Let $x_1, \dots, x_n$ be pairwise distinct elements of $L_1$ and $u_1, \dots, u_m$ be pairwise distinct elements of $L_2$. Then $\vdash^n \Gamma$ implies $\vdash^n \Gamma\{y_1/x_1, \dots, y_n/x_n, v_1/u_1, \dots, v_m/u_m\}$ for all $y_1, \dots, y_n \in L_1$ and $v_1, \dots, v_m \in L_2$.*

(iii) (weakening) $\vdash^n \Gamma \Longrightarrow \vdash^n \Gamma, J$ *for every judgement $J$*

(iv) ($R$-contraction) $\vdash^n \Gamma, u\,\overline{R}\,v, u\,\overline{R}\,v \Longrightarrow \vdash^n \Gamma, u\,\overline{R}\,v$

(v) ($W$-contraction)
 (a) $\vdash^n \Gamma, x\,\overline{W}\,u, x\,\overline{W}\,u \Longrightarrow \vdash^n \Gamma, x\,\overline{W}\,u$
 (b) $\vdash^n \Gamma, x\,\overline{W}\,u, (x, u){:}\ A \Longrightarrow \vdash^n \Gamma, (x, u){:}\ A$

(vi) ($R$-reflexivity) $\vdash^n \Gamma, u\,\overline{R}\,u \Longrightarrow \vdash^n \Gamma$

**Proof.** Straightforward induction on the height of the given derivation. The proof of Facts (ii),(iii) need Fact (i) in order to avoid a clash with eigenvariables. Intuitively, Fact (v)(b) holds because $(x, u) : A$ is treated throughout as if it were of the form $x\,\overline{W}\,u \vee \dots$ and we can contract several occurrences of $x\,\overline{W}\,u$. Technically, we use the fact that the $j(x, u)$-condition can also be fulfilled by an occurrence of $(x, u) : A$, not only $x\,\overline{W}\,u$, and that an occurrence of $x\,\overline{W}\,u$ can be contracted into the constructed formula $\Box B$ or $\mathrm{K}B$ in $\Box/\mathrm{K}$-inferences. In the cases where $A$ is constructed by $\Box, \mathrm{K}$ we use (v)(a) in the proof. Furthermore, Facts (iv) and (v) are used in the proof of (vi) in the case of frame rules. □

With Fact (i), we can always rename eigenvariables in a proof so that they become different from each other and from every other variable in a given judgement, sequent or second derivation. We make use of this fact in many of the proofs below without mentioning it explicitly.

**Lemma 5.2** *The **HSS**-rules of necessitation are admissible.*

**Proof.** Let $(x, u)\colon A$ be derivable. Then, by label substitution, also $(x, v)\colon A$ and $(y, u)\colon A$ are derivable for fresh $v \in L_2, y \in L_1$. By the admissibility of weakening we obtain the derivability of $x\,\overline{W}\,u\,,\ u\,\overline{R}\,v\,,\ (x, v)\colon A$ and of $x\,\overline{W}\,u\,,\ (y, u)\colon A$. Application of ($\Box$) or ($\textsc{k}$) respectively yields a derivation of $(x, u)\colon \Box A$ or $(x, u)\colon \textsc{k}A$. □

For ($\Diamond$) and ($\textsc{l}$), inversion is just an instance of weakening. So the theorem below yields the height-preserving invertibility of all logical rules.

**Theorem 5.3 (Invertibility of logical rules)** *The following holds for the calculus **LSSL-p**:*

(i) ($\wedge$-inversion) $\vdash^n \Gamma, (x, u)\colon A \wedge B \Longrightarrow \vdash^n \Gamma, (x, u)\colon A$ *and* $\vdash^n \Gamma, (x, u)\colon B$

(ii) ($\vee$-inversion) $\vdash^n \Gamma, (x, u)\colon A \vee B \Longrightarrow \vdash^n \Gamma, (x, u)\colon A, (x, u)\colon B$

(iii) ($\Box$-inversion) $\vdash^n \Gamma, (x, u)\colon \Box A \Longrightarrow \vdash^n \Gamma, u\,\overline{R}\,v, x\,\overline{W}\,u, (x, v)\colon A$ *for every* $v \in L_2$

(iv) ($\textsc{k}$-inversion) $\vdash^n \Gamma, (x, u)\colon \textsc{k}A \Longrightarrow \vdash^n \Gamma, x\,\overline{W}\,u, (y, u)\colon A$ *for all* $y \in L_1$

**Proof.** By induction on the height of the derivation. In the proof of Fact (iii) and (iv), the additional $x\,\overline{W}\,u$ is used in the case where $(x, u)\colon \Box A$ or $(x, u)\colon \textsc{k}A$ is necessary to meet the context condition $j(x, u)$ for the last inference in the given derivation. □

**Theorem 5.4 (Admissibility of contraction)** *If* $\vdash^n \Gamma, (x, u)\colon A, (x, u)\colon A$ *then* $\vdash^n \Gamma, (x, u)\colon A$

**Proof.** By induction on the height of the derivation. In the case that one of the distinguished occurrences of $(x, u)\colon A$ is constructed by the last inference (and the principle symbol of $A$ is not $\Diamond$ or $\textsc{l}$), we combine inversion with the induction hypothesis. In the case of ($\Box$), i.e.:

$$(\Box)\ \frac{\dfrac{\vdots}{\Gamma\,,\ (x, u)\colon \Box A\,,\ x\,\overline{W}\,u\,,\ u\,\overline{R}\,v\,,\ (x, v)\colon A}}{\Gamma\,,\ (x, u)\colon \Box A\,,\ (x, u)\colon \Box A}\ !(v)$$

we use height-preserving inversion of ($\Box$) first, then the IH, followed by ($R$-contraction) and ($W$-contraction), and finally an application of ($\Box$) to build the desired derivation. The case of ($\textsc{k}$) is similar, without ($R$-contraction). □

Next we demonstrate the strength of the calculus by presenting derivations for the **HSS** axioms. With negation defined as above, this is provided by the derivability of the sequents (ii)-(ix) in the next lemma.

**Lemma 5.5** *For all SSL-formulas $A, B$, predicate letters $P$, $x \in L_1$, and $u \in L_2$, the following sequents are derivable in **LSSL-p**:*

(i) $(x,u)\colon A\,,\ (x,u)\colon \neg A$

(ii) $(x,u)\colon A$ *where $A$ is a substitution instance of a tautology of classical propositional logic*

(iii) $(x,u)\colon (\neg P \vee \Box P) \wedge (P \vee \Box\neg P)$ *for propositional letters $P$*

(iv) $(x,u)\colon \textsc{l}\neg A\,,\ (x,u)\colon A \wedge \textsc{kk}A$

(v) $(x,u)\colon \Diamond\neg A\,,\ (x,u)\colon A \wedge \Box\Box A$

(vi) $(x,u)\colon \textsc{l}(A \wedge \neg B)\,,\ (x,u)\colon \textsc{l}\neg A\,,\ (x,u)\colon \textsc{k}B$

(vii) $(x,u)\colon \Diamond(A \wedge \neg B)\,,\ (x,u)\colon \Diamond\neg A\,,\ (x,u)\colon \Box B$

(viii) $(x,u)\colon \textsc{k}\neg A\,,\ (x,u)\colon \textsc{kl}A$

(ix) $(x,u)\colon \textsc{l}\Diamond\neg A\,,\ (x,u)\colon \Box\textsc{k}A$

**Proof.** For Fact (i), we proceed by straightforward induction on $A$ as usual. One can obtain a derivation as required in Fact (ii) from a cut-free derivation of the tautology in **GS3**. We present the derivation of sequent (ix), the cross axiom:

$$\dfrac{\text{Fact (i), weakening}}{(x,u)\colon \textsc{l}\Diamond\neg A, (y,u)\colon \Diamond\neg A, (y,v)\colon \neg A, x\,\overline{W}\,u, x\,\overline{W}\,v, y\,\overline{W}\,u, (y,v)\colon A, u\,\overline{R}\,v}$$

$$(\Diamond)\ \dfrac{(x,u)\colon \textsc{l}\Diamond\neg A, (y,u)\colon \Diamond\neg A, (y,v)\colon \neg A, x\,\overline{W}\,u, x\,\overline{W}\,v, y\,\overline{W}\,u, (y,v)\colon A, u\,\overline{R}\,v}{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ (y,u)\colon \Diamond\neg A\,,\ x\,\overline{W}\,u\,,\ x\,\overline{W}\,v\,,\ y\,\overline{W}\,u\,,\ (y,v)\colon A\,,\ u\,\overline{R}\,v}$$

$$(\textsc{l})\ \dfrac{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ (y,u)\colon \Diamond\neg A\,,\ x\,\overline{W}\,u\,,\ x\,\overline{W}\,v\,,\ y\,\overline{W}\,u\,,\ (y,v)\colon A\,,\ u\,\overline{R}\,v}{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ x\,\overline{W}\,u\,,\ x\,\overline{W}\,v\,,\ y\,\overline{W}\,u\,,\ (y,v)\colon A\,,\ u\,\overline{R}\,v}$$

$$(RW)\ \dfrac{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ x\,\overline{W}\,u\,,\ x\,\overline{W}\,v\,,\ y\,\overline{W}\,u\,,\ (y,v)\colon A\,,\ u\,\overline{R}\,v}{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ x\,\overline{W}\,u\,,\ x\,\overline{W}\,v\,,\ (y,v)\colon A\,,\ u\,\overline{R}\,v}$$

$$(\textsc{k})\ \dfrac{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ x\,\overline{W}\,u\,,\ x\,\overline{W}\,v\,,\ (y,v)\colon A\,,\ u\,\overline{R}\,v}{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ x\,\overline{W}\,u\,,\ (x,v)\colon \textsc{k}A\,,\ u\,\overline{R}\,v}\ !(y)$$

$$(\Box)\ \dfrac{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ x\,\overline{W}\,u\,,\ (x,v)\colon \textsc{k}A\,,\ u\,\overline{R}\,v}{(x,u)\colon \textsc{l}\Diamond\neg A\,,\ (x,u)\colon \Box\textsc{k}A}\ !(v)$$

The remaining derivations are given in the appendix. □

## 6 Admissibility of cut

**Lemma 6.1** *If $\vdash^n \Gamma\,,\ (x,u)\colon A$ and $\vdash^m \Pi\,,\ (x,u)\colon \neg A$, then $\vdash \Gamma, \Pi\,,\ xWu$.*

**Proof.** By induction on $A$, side induction on $n+m$.

*Case 1:* $(x,u)\colon A$ is of no relevance for the last inference in the first of the given derivations, or $(x,u)\colon \neg A$ is of no relevance for the last inference in the second derivation. If the corresponding conclusion is an axiom then $\Gamma, \Pi$ is an axiom. Otherwise, use the side induction hypothesis. (Rename eigenvariables first if necessary.)

*Case 2:* $(x,u)\colon A$ is relevant for the last inference but only to meet the context condition $j(x,u)$:

$$\dfrac{\Gamma'\,,\ (x,u)\colon A}{\Gamma\,,\ (x,u)\colon A}$$

By side induction hypothesis we obtain $\Gamma', \Pi, x\,\overline{W}\,u$, from which we can deduce

$\Gamma, \Pi, x\,\overline{W}\,u$.

*Case 3:* $(x,u)\colon \neg A$ is relevant for the last inference but only to meet the context condition $j(x,u)$: similar

*Case 4:* $(x,u)\colon A$ and $(x,u)\colon \neg A$ are the principal formulas in the last inference of the respective derivations. Then we distinguish cases according to $A$.

If $A$ is a literal, then $\Gamma, (x,u)\colon A$ and $\Pi, (x,u)\colon \neg A$ are axioms with principal formulas $(x,u)\colon A$ and $(x,u)\colon \neg A$ respectively. In that case $\Gamma$ contains $(x,v)\colon \neg A$ for some $v$ and $\Pi$ contains $(x,w)\colon A$ for some $w$. Hence $\Gamma, \Pi$ is an axiom.

Furthermore, we present in detail the case of the principal symbols $\Box/\Diamond$ in $A/\neg A$. The remaining cases are similar, even a bit simpler.

W.l.o.g. $A \equiv \Box B$ and $\neg A \equiv \Diamond\neg B$. Then the derivations have the form:

$$d_1: \quad (\Box)\,\frac{\dfrac{\vdots}{\Gamma,\ x\,\overline{W}\,u\,,\ u\,\overline{R}\,v\,,\ (x,v)\colon A}}{\Gamma\,,\ (x,u)\colon \Box A}\,!(v)$$

and

$$d_2: \quad (\Diamond/\Diamond_{ref})\,\frac{\dfrac{\vdots}{\Pi'\,,\ (u\,\overline{R}\,w\,,\ )(x,u)\colon \Diamond\neg A\,,\ (x,w)\colon \neg A}}{\Pi'\,,\ (u\,\overline{R}\,w\,,\ )(x,u)\colon \Diamond A}\,j(x,w)$$

where $u\,\overline{R}\,w$ may be missing if $u = w$, and $\Pi = \Pi', u\,\overline{R}\,w$ otherwise. First we obtain $\vdash\ \Gamma, \Pi', (u\,\overline{R}\,w\,,\ )(x,w)\colon \neg A\,,\ x\,\overline{W}\,u$ by side induction hypothesis. Second, applying substitution to the immediate subderivation of $d_1$, we obtain a derivation of $\Gamma\,,\ x\,\overline{W}\,u\,,\ u\,\overline{R}\,w\,,\ (x,w)\colon A$. Combining these and using the (main) induction hypothesis, we get:

$$\vdash \Gamma\,,\ x\,\overline{W}\,u\,,\ u\,\overline{R}\,w\,,\ \Gamma\,,\ \Pi'\,,\ (u\,\overline{R}\,w\,,\ )\,,\ x\,\overline{W}\,u\,,\ x\,\overline{W}\,w$$

*Subcase 1:* The inference introducing $\neg A \equiv \Diamond\neg B$ was $\Diamond_{ref}$. Then the second $u\,\overline{R}\,w$ is missing, $\Pi' = \Pi$ and $u = w$. In that case, we use the admissibility of the reflexivity rule and (W-contraction) to obtain $\vdash \Gamma\,,\ x\,\overline{W}\,u\,,\ \Gamma\,,\ \Pi$.

*Subcase 2:* Otherwise. Then $\Pi = \Pi', u\,\overline{R}\,w$ and $\Pi$ contains $x\,\overline{W}\,w$ or a judgement of the form $(x,w)\colon B$. Now we apply (R-contraction), (W-contraction) of type (a), and (W-contraction) of type (a) or (b) to obtain again $\vdash \Gamma\,,\ x\,\overline{W}\,u\,,\ \Gamma\,,\ \Pi$.

In both cases, the proof is completed by applying (contraction) for the formulas in $\Gamma$. $\Box$

**Corollary 6.2** *Let* $\Gamma, \Pi$ *be sequents so that* $\Gamma, \Pi$ *contains* $x\,\overline{W}\,u$ *or some judgement of the form* $(x,u)\colon B$*. If* $\vdash \Gamma\,,\ (x,u)\colon A$ *and* $\vdash \Pi\,,\ (x,u)\colon \neg A$*, then* $\vdash \Gamma, \Pi$.

**Proof.** Use 6.1 and (W-contraction). $\Box$

**Lemma 6.3** *The rule* modus ponens *is admissible.*

**Proof.** Let $(x,u)\colon A \to B$ and $(x,u)\colon A$ be derivable. By $(\vee)$-inversion, the sequent $(x, u)\colon \neg A$, $(x, u) : B$ is also derivable. Using 6.2, we can conclude that $(x, u)\colon B$ is derivable. □

**Lemma 6.4** *If **HSS** $\vdash A$ then **LSSL-p** $\vdash (x, u)\colon A$ for all $x \in L_1, u \in L_2$.*

**Proof.** By induction on the length of a derivation in **HSS**, using 5.5, 5.2 and 6.3. □

As it has been proved in [1] that **HSS** is complete w.r.t. subset spaces, we have the following theorem:

**Theorem 6.5** ***LSSL-p** is complete w.r.t. abstract subset spaces.*

In the next section we present a direct proof of this result based on proof search in **LSSL-p**.

## 7 Direct proof of completeness

In this section, we present the definition of a *search tree* which reflects bottom-up proof search for **LSSL-p** and use it for the proof of completeness. Completeness proofs in this style for labelled calculi have been presented before e.g. in [7,5,16]. Our calculus, however, does not belong to the class considered in [7,5], and we can not expect to construct a *finite* countermodel, as we are working with (abstract) subset spaces. In particular, we will not obtain a *decision procedure*. However, we get a reasonable strategy for the construction of proofs, as evidence for validity, which produces an output for all valid formulas. To simplify a our procedure, we consider a ***c**ompressed* version of derivations and a corresponding system **LSSL-pc**:

- *Frame rules* are never applied explicitly. We reformulate the application conditions instead. To this end, we use the relations $\mathbf{R}, \mathbf{W}$ defined in Sec. 4.2. on page 278.
- All reductions corresponding to *disjunctive* rules, i.e. rules introducing $\vee, \Diamond, \textsc{l}$ are performed in one step. For this, we let $\mathbf{D}(\Gamma)$ denote the least multiset extending $\Gamma$ that satisfies:
  - · $(x, u)\colon A$ and $(x, u)\colon B$ are in $\mathbf{D}(\Gamma)$ if $(x, u)\colon A \vee B$ is in $\mathbf{D}(\Gamma)$.
  - · $(x, v)\colon A$ is in $\mathbf{D}(\Gamma)$ if $(x, v) \in \mathbf{W}(\Gamma)$ and $(x, u)\colon \Diamond A$ is in $\mathbf{D}(\Gamma)$ for some $u$ so that $(u, v) \in \mathbf{R}(\Gamma)$
  - · $(y, u)\colon A$ is in $\mathbf{D}(\Gamma)$ if $(y, u) \in \mathbf{W}(\Gamma)$ and $(x, u)\colon \textsc{l}A$ is in $\mathbf{D}(\Gamma)$ for some $x$

Note that in the definition of $\mathbf{R}(\Gamma)$ we build the reflexive-transitive closure *on* $\boldsymbol{L}_2(\Gamma)$. As a consequence, the sets $\mathbf{R}(\Gamma)$ and $\mathbf{W}(\Gamma)$ are finite. Obviously, $\mathbf{D}(\Gamma)$ is also finite and can be computed in a straightforward way. Applying the rule ($R$-trans) from bottom to top, we can turn every $\Gamma$ into a sequent which contains $u\,\overline{R}\,v$ for every pair in $\mathbf{R}_0(\Gamma)^+$. With the help of $(RW)$ we can then obtain a sequent which satisfies the $j(x, v)$-condition for all pairs $(x, v)$ in $\mathbf{W}(\Gamma)$. With this in mind, we see that a derivation of $\mathbf{D}(\Gamma)$ can be turned into a derivation of $\Gamma$ by applications of (weakening), (contraction), disjunctive rules (i.e. $(\Diamond), (\Diamond_{ref}), (\textsc{l}), (\vee)$) and frame rules.

$$(\wedge)\ \frac{\Gamma, (x,u)\colon A \qquad \Gamma, (x,u)\colon B}{\Gamma, (x,u)\colon A \wedge B} \qquad (\mathrm{ax})\ \frac{}{\Gamma, (x,u)\colon P, (x,v)\colon \neg P} \qquad (\mathbf{D})\ \frac{\mathbf{D}(\Gamma)}{\Gamma}$$

$$(\Box)\ \frac{\Gamma, x\,\overline{W}\,u, u\,\overline{R}\,v, (x,v)\colon A}{\Gamma, (x,u)\colon \Box A}\ !(v) \qquad (\mathrm{K})\ \frac{\Gamma, x\,\overline{W}\,u, (y,u)\colon A}{\Gamma, (x,u)\colon \mathrm{K}A}\ !(y)$$

Table 4
System **LSSL-pc**

Now we consider trees (finite and infinite) of sequents built according to the rules in Table 4. We write seq$(N)$ for the sequent at node $N$.

The rules $(\wedge), (\Box), (\mathrm{K})$ and their constructed principal formulas as well as the corresponding judgements are called *conjunctive*. An *expansion step* for such a formula consists of adding nodes with the corresponding premisses as children. The expansion step for a judgement $(x,u)\colon A \wedge B$ at leaf $N$ will be performed only if neither $(x,u)\colon A$ nor $(x,u)\colon B$ occurs on the path $\alpha(N)$ leading from the root to $N$. Similarly, the step for $(x,u)\colon \Box A$ at $N$ will be performed only if no judgement $(x,v)\colon A$ satisfying $(u,v) \in \mathbf{R}(\Gamma)$ occurs on $\alpha(N)$, and the expansion for $(x,u)\colon \mathrm{K}A$ is subject to the restriction that no judgement $(y,u)\colon A$ occurs on $\alpha(N)$. Furthermore, we apply these steps only to nodes which are no axioms.

Applying these steps successively to all conjunctive judgements at leaves that meet these conditions (but not to conjunctive formulas produced by this transformation) is called a *C-step* for the tree $T$. A *D-step* for it consists in adding a child $N'$ to every leaf $N$ and let seq$(N') := \mathbf{D}(\text{seq}(N))$.

To search for a derivation of $(x,u)\colon A$, we proceed as follows: Start with the one-node-tree with $(x,u)\colon A$ at its root, and add a single child $N$ with seq$(N) = \mathbf{D}((x,u)\colon A)$. As long as there are still leaves $N$ so that seq$(N)$ is no axiom and all thoses leaves are expandable, perform a C-step followed by a D-step. Now, if this procedure terminates with a tree where all leaves contain axioms, then we have found a derivation of $(x,u)\colon A$ in **LSSL-pc** which can be uncompressed to give a derivation of $(x,u)\colon A$ in **LSSL-p**. If the procedure terminates with a leaf with no axiom that is not expandable, then we consider the path $\alpha$ from the root to that node, and let $(\Gamma_i)_{i\in I}$ with appropriate $I = \{0, \ldots, n\}$ be the corresponding sequence of sequents. Otherwise, the procedure generates an infinite tree which contains an infinite branch $\alpha$, and we let $\alpha = (\Gamma_i)_{i\in I}$ with $I = \mathbb{N}$ denote the infinite sequence of corresponding sequents. In both cases, we let $\Gamma := \bigcup_{i\in I} \Gamma_i$.

Now it remains to be shown that we can obtain a countermodel based on a path $\alpha$ as described above. To this end, we extend the definitions in Sec. 4.2, page 278, to *infinite* $\Gamma$, and let $X := \mathbf{L}_1(\Gamma)$, $\mathcal{O} := \mathbf{L}_2(\Gamma)$, $\mathcal{W}(\Gamma) := \mathbf{W}(\Gamma)$, and $\mathcal{R} := \mathbf{R}(\Gamma)$. Then $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ is the abstract subset frame generated by $(X, \mathcal{O}, \mathbf{W}_0(\Gamma), \mathbf{R}_0(\Gamma))$.

Due to the construction of the tree, judgements $(x,u)\colon P$ or $(x,u)\colon \neg P$ for $P \in PV$ occur in every $\Gamma_j$ with $j \geq i$ if they occur in $\Gamma_i$, and we know that no $\Gamma_i$ is an axiom. As a consequence, there is a valuation $\mathcal{V}$ for $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ so that

$$\mathcal{V}(y)(P) = \mathbf{t} \text{ if } (y,v)\colon \neg P \text{ occurs in } \alpha \text{ for some } v \in L_2$$
$$\mathcal{V}(y)(P) = \mathbf{f} \text{ if } (y,v)\colon P \text{ occurs in } \alpha \text{ for some } v \in L_2$$

Let $\mathcal{M} = (X, \mathcal{O}, \mathcal{W}, \mathcal{R}, \mathcal{V})$, choose $\ell_1, \ell_2$ so that $\ell_1(y) = y$ for all $y \in L_1$ and $\ell_2(v) = v$ for all $v \in L_2$.

The construction of the tree also ensures that judgements $u\,\overline{R}\,v$, $x\,\overline{W}\,u$, $(x,u)\colon A \vee B$, $(x,u)\colon \Diamond A$, $(x,u)\colon \textsc{l}A$ occur in every $\Gamma_j$ with $j \geq i$ if they occur in $\Gamma_i$. Furthermore, if some $\Gamma_i$ satisfies the $j(x,u)$-condition, then so does every $\Gamma_j$ with $j \geq i$. This is used in proving the following facts:

(i) $\mathbf{R}(\Gamma_{i-1}) \subseteq \mathbf{R}(\Gamma_i)$ for all $i \in \mathbb{N}_+ \cap I$ and $\mathcal{R} = \bigcup_{i \in I} \mathbf{R}(\Gamma_i)$

(ii) $\mathbf{W}(\Gamma_{i-1}) \subseteq \mathbf{W}(\Gamma_i)$ for all $i \in \mathbb{N}_+ \cap I$ and $\mathcal{W} = \bigcup_{i \in I} \mathbf{W}(\Gamma_i)$

(iii) If $(y,v)\colon B \wedge C$ in $\Gamma$ then $(y,v)\colon B$ in $\Gamma$ or $(y,v)\colon C$ in $\Gamma$

(iv) If $(y,v)\colon B \vee C$ in $\Gamma$ then $(y,v)\colon B$ in $\Gamma$ and $(y,v)\colon C$ in $\Gamma$

(v) If $(y,v)\colon \Diamond B$ in $\Gamma$ then $(y,w)\colon B$ in $\Gamma$ for all $w$ so that $(v,w) \in \mathcal{R}$ and $(y,w) \in \mathcal{W}$.

(vi) If $(y,v)\colon \textsc{l}B$ in $\Gamma$ then $(z,v)\colon B$ in $\Gamma$ for all $z$ so that $(z,v) \in \mathcal{W}$.

(vii) If $(y,v)\colon \Box B$ in $\Gamma$ then there is $w$ so that $(v,w) \in \mathcal{R}$ and $(y,w)\colon B$ in $\Gamma$ (and hence also $(y,w) \in \mathcal{W}$).

(viii) If $(y,v)\colon \textsc{k}B$ in $\Gamma$ then there is $z$ so that $(z,v)\colon B$ in $\Gamma$ (and hence also $(z,v) \in \mathcal{W}$).

To see Fact (viii), assume that $\textsc{k}B$ is in $\Gamma$. Then at some point of the construction, the tree had a node of $\alpha$ containing $\textsc{k}B$ as a leaf and the corresponding expansion was considered. Either the expansion was performed, in which case the next node of $\alpha$ contains some $(z,v)\colon B$, or the expansion was rejected. If it was rejected then there is some $(z,v)\colon B$ earlier in $\alpha$.

We present the proof of Fact (v) as an example for the disjunctive connectives: Let $(y,v)\colon \Diamond B$ be in $\Gamma_i$, $(v,w) \in \mathcal{R}$ and $(y,w) \in \mathcal{W}$. By Facts (i) and (ii) we get $j,k \in I$ so that $(v,w) \in \mathbf{R}(\Gamma_j)$ and $(y,w) \in \mathbf{W}(\Gamma_k)$. Now let $m \geq \max\{i,j,k\}$ so that $\Gamma_{m+1} = \mathbf{D}(\Gamma_m)$. Then $(y,v)\colon B$ is in $\Gamma_{m+1}$.

The remaining cases are similar. With these facts, an easy induction on $B$ shows that $(\mathcal{M}, \ell_1, \ell_2) \not\models (y,v)\colon B$ for all $(y,v)\colon B$ in $\alpha$, in particular $\mathcal{M} \not\models (x,u)\colon A$.

## 8 Conclusion and further work

The logic of subset spaces is a bimodal logic with no obvious, pure, cut-free sequent system. The labelled approach, however, offers an alternative. Introducing a new variant which makes use of compound expressions for worlds and an "is-world"-predicate, we obtain a solution which is satisfactory in several ways: it is close to the original semantics and it uses only simple frame rules. The completeness proof is based on proof search, producing a (possibly infinite) countermodel for non-valid formulas.

The cut-free sequent calculus **LSSL-p** has been developed as a contribution to the *proof theory* of subset space logic. It has the *subformula* and *separation property*, and it provides a promising starting point for further proof-theoretic investigations of this logic and its extensions. If the focus is on *automatic deduction*, other types of calculi may be preferable. The study of tableau systems for SSL based on cross axiom models is current work of others. However, the development of termination conditions for **LSSL-p** could yield an interesting candidate for comparison.

Based on subset space logic, *topologic* is also introduced in [1]. A natural continuation of our work would be an extension to that system.

The labelled approach is sometimes criticised for not being *purely proof-theoretic*, as the semantics is internalised in the system. Although it constitutes a very convincing method for constructing calculi in difficult cases as SSL, the quest for a fine, "pure" system remains interesting. The calculus **LSSL-p** presented here provides new insights which may prove helpful also for this goal. However, even if alternatives will be developed, the labelled **LSSL-p** is still a rather simple and natural solution.

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# Appendix

## A Derivability of the HSS axioms

In most of the derivations below we use Fact 1 in 5.3 plus weakening.

Cross axiom: see Section 5; Persistence: Let $\Pi :\equiv u\,\overline{R}\,v\,,\ x\,\overline{W}\,u$ in:

$$(\wedge)\,\dfrac{(\vee)\,\dfrac{(\Box)\,\dfrac{(ax)\,\dfrac{}{(x,u)\colon \neg P\,,\ (x,v)\colon P\,,\ \Pi}}{(x,u)\colon \neg P\,,\ (x,u)\colon \Box P}\,!(v)}{(x,u)\colon \neg P \vee \Box P} \qquad \dfrac{\dfrac{\dfrac{}{(x,u)\colon P\,,\ (x,v)\colon \neg P\,,\ \Pi}\,(ax)}{(x,u)\colon P\,,\ (x,u)\colon \Box\neg P}\,!(v)}{(x,u)\colon P \vee \Box\neg P}\,(\vee)}{(x,u)\colon (\neg P \vee \Box P) \wedge (P \vee \Box\neg P)}$$

T and K4 for $\Box$:

$$(\Diamond_{ref})\,\dfrac{(x,u)\colon \Diamond\neg A\,,\ (x,u)\colon \neg A\,,\ (x,u)\colon A}{(x,u)\colon \Diamond\neg A\,,\ (x,u)\colon A}$$

$$(\Box)\,\dfrac{(\Box)\,\dfrac{(R\text{-trans})\,\dfrac{(\Diamond)\,\dfrac{(x,u)\colon \Diamond\neg A\,,\ (x,w)\colon \neg A\,,\ (x,w)\colon A\,,\ x\,\overline{W}\,v\,,\ v\,\overline{R}\,w\,,\ x\,\overline{W}\,u\,,\ u\,\overline{R}\,v\,,\ u\,\overline{R}\,w}{(x,u)\colon \Diamond\neg A\,,\ (x,w)\colon A\,,\ x\,\overline{W}\,v\,,\ v\,\overline{R}\,w\,,\ x\,\overline{W}\,u\,,\ u\,\overline{R}\,v\,,\ u\,\overline{R}\,w}}{(x,u)\colon \Diamond\neg A\,,\ (x,w)\colon A\,,\ x\,\overline{W}\,v\,,\ v\,\overline{R}\,w\,,\ x\,\overline{W}\,u\,,\ u\,\overline{R}\,v}\,!(\mathrm{w})}{(x,u)\colon \Diamond\neg A\,,\ (x,v)\colon \Box A\,,\ x\,\overline{W}\,u\,,\ u\,\overline{R}\,v}\,!(\mathrm{v})}{(x,u)\colon \Diamond\neg A\,,\ (x,u)\colon \Box\Box A}$$

Normality for $\Box$: Let $\Gamma :\equiv (x,u)\colon \Diamond(A\wedge\neg B)\,,\ (x,u)\colon \Diamond\neg A\,,\ x\,\overline{W}\,u$ in:

$$(\Box)\,\dfrac{(\Diamond)\,\dfrac{(\wedge)\,\dfrac{(\Diamond)\,\dfrac{\Gamma, (x,v)\colon A, (x,v)\colon \neg A, (x,v)\colon B, u\,\overline{R}\,v}{\Gamma, (x,v)\colon A, (x,v)\colon B, u\,\overline{R}\,v} \qquad \Gamma, (x,v)\colon \neg B, (x,v)\colon B, u\,\overline{R}\,v}{\Gamma, (x,v)\colon A\wedge\neg B, (x,v)\colon B, u\,\overline{R}\,v}}{\Gamma, (x,v)\colon B, u\,\overline{R}\,v}}{(x,u)\colon \Diamond(A\wedge\neg B)\,,\ (x,u)\colon \Diamond\neg A, (x,u)\colon \Box B}\,!(v)$$

The corresponding properties for K are proved similar.
Euclidean property:

$$
\dfrac{\dfrac{\dfrac{x\,\overline{W}\,u\,,\,(y,u)\colon \neg A\,,\,(z,u)\colon \textsc{l}A\,,\,(y,u)\colon A\,,\,x\,\overline{W}\,u}{x\,\overline{W}\,u\,,\,(y,u)\colon \neg A\,,\,(z,u)\colon \textsc{l}A\,,\,x\,\overline{W}\,u}\,(\textsc{l})}{(y,u)\colon \neg A\,,\,(x,u)\colon \textsc{kl}A\,,\,x\,\overline{W}\,u}\,(\textsc{k})\;!(z)}{(x,u)\colon \textsc{k}\neg A\,,\,(x,u)\colon \textsc{kl}A}\,(\textsc{k})\;!(y)
$$

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