Logics of Infinite Depth

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Abstract

Consider a definition of depth of a logic as the supremum of ordinal types of well-ordered descending chains. This extends the usual definition of codimension to infinite depths. Logics may either have no depth, or have countable depth in case a maximal well-ordered chain exists, or be of depth ω_1. We shall exhibit logics of all three types. We show in particular that many well-known systems, among them K, $\mathsf{K4}$, G, Grz and $\mathsf{S4}$, have depth ω_1. Basically, if a logic is the intersection of its splitting logics and has finite model property, then either the splitting logics have an infinite antichain (and the depth is therefore ω_1) or the splitting logics form a well-partial order whose supremum type is realised and therefore countable, though it may be different from the supremum type of the splitting logics alone.

Keywords: Lattices of Modal Logics, Well Partial Orders, Splittings

1 Introduction

The lattice Ext K is very complex. It contains continuously many complete logics. Moreover, completeness is a rather rare property. Early results by Blok revealed that complete logics above K are either intrinsically complete, that is, they are the only logics having the same class of Kripke-frames; or there are $2^{\aleph_0}$ logics with the same class of Kripke-frames. Most known systems are of the latter kind. The so-called *degree of incompleteness* is thus either 1 or of size continuum. The largest of these (which is the only complete logic in the spectrum) has $2^{\aleph_0}$ cocovers. So, order and chaos are quite close together.

In order to understand the structure of the lattice Ext K one may either study some special systems, or obtain some rough outline of the global structure. There are many ways to go. From the abovementioned results it emerged that the notion of a splittings is important for studying the structure of lattices of logics. However, even this structure is sometimes very complex and therefore some reduction in complexity may be useful. In [5] we have for example looked at groups of automorphisms of the lattices Ext L for various L. Obviously, the larger the group the more homogeneous and simpler the lattice. Alternatively, one may study some numerical invariants of logics such as depth. These topics are linked. Clearly, if Ext L admits an automorphism then that automorphism must leave the depth of logics invariant. However, there is a problem in that depth is typically a cardinal number and therefore of limited usefulness. Most interesting logics have infinite depth, and that depth is countable.

Thus we have decided to look at a definition of depth that yields an ordinal number instead. We define the (*ordinal*) *depth* of a logic L as the supremum of all κ such that there exists a downgoing chain of order type $\kappa + 1$ of logics above L such that $L = L_\kappa$. Here, a sequence $\langle L_\lambda : \lambda < \mu \rangle$ is a *downgoing chain* if for all $\lambda < \mu$ either (i) $\lambda = \lambda' + 1$ and L_λ is immediately below $L_{\lambda'}$, or (ii) λ is a limit ordinal and $L_\lambda = \bigsqcap_{\lambda' < \lambda} L_{\lambda'}$. In the finite case this equals the standard definition of depth or codimension. (L has codimension n in lattice theoretic terms iff there is a downgoing chain $L_0 \succ L_1 \succ \cdots \succ L_n = L$, starting at the top, that is, $L_0 = \mathsf{K} \oplus \bot$. This chain has order type $n+1$. By definition, L thus has depth n.) We shall see that there exist logics of varying infinite depth. For a start we note that a logic has depth 0 iff it is the inconsistent logic $\mathsf{K} \oplus \bot$. Obviously, the ordinal depth is invariant under automorphisms.

If the extension lattice of a logic has an antichain of size $\aleph_0$ then the order type can grow up to ω_1, the first uncountable ordinal. We establish this for a number of logics, including $\mathsf{S4}$, G, $\mathsf{K4}$ and K. If however the lattice has only finite antichains, chances are that it is a continuous lattice. Its depth is then a countable ordinal, and in that case the results by [3] can be applied to give exact bounds for the depth of the logic.

Notice that the order type of upgoing chains is generally different. For example, Ext K has no atoms, hence there is no well-ordered upgoing chain starting at K. The same applies to many other logics. [1]

2 Preliminaries

The lattice operations are denoted by $\sqcap$ and $\sqcup$ (and their infinitary versions by $\bigsqcap$ and $\bigsqcup$). x is a *lower cover* or *cocover* of y, in symbols $x \prec y$, if there is no z such that $x < z < y$. x has *dimension n over y* (and y has *codimension n under x*) if there is a finite chain $y = y_0 \prec y_1 \prec y_2 \prec \cdots \prec y_n = x$. In a modular lattice, any such chain has the same length, so the number does not depend on the choice of the sequence.

The lattice Ext K—in general any lattice Ext L of normal extensions of a modal logic L—is a locale, that is, a complete and distributive lattice that enjoys the following infinitary distributive law (see [6]).

$$L \sqcap \bigsqcup_{i \in I} M_i = \bigsqcup_{i \in I} (L \sqcap M_i) \tag{1}$$

[1] This paper is dedicated to the memory of Alexander Chagrov. I also wish to thank Stefan Geschke for his help.

L is called *continuous* if in addition

$$L \sqcup \bigsqcap_{i \in I} M_i = \bigsqcap_{i \in I} (L \sqcup M_i) \tag{2}$$

The law (2) is *not* valid in Ext K. However, for certain logics L the lattive Ext L is in fact continuous. An example is $\mathsf{S4.3}$. A logic L is $\bigsqcap$-*prime* if for every family of logics such that $\bigsqcap_{i \in I} M_i \leqslant L$ there is an $i \in I$ such that $M_i \leqslant L$. Dually for $\bigsqcup$-*prime.* L is $\bigsqcap$-*irreducible*, if for every family of logics such that $\bigsqcap_{i \in I} M_i = L$ there is an $i \in I$ such that $M_i = L$. Dually for $\bigsqcup$-irreducible. A logic is $\bigsqcup$-irreducible iff it is $\bigsqcup$-prime. Indeed, suppose that L is $\bigsqcup$-irreducible and let M_i, $i \in I$, be logics such that $\bigsqcup_{i \in I} M_i \geqslant L$. Then $L = L \sqcap \bigsqcup_{i \in I} M_i = \bigsqcup_{i \in I} L \sqcap M_i$, so there is an $i \in I$ such that $L = L \sqcap M_i$, which implies that $L \leqslant M_i$. Every $\bigsqcap$-prime logic is also $\bigsqcap$-irreducible, but the converse does not hold. This is because (2) fails to hold. (The logic of the one-point reflexive frame is a case in point.)

Prime logics are related to *splittings.* A *splitting* of a lattice $\mathcal{L} = \langle L, \sqcap, \sqcup \rangle$ is a disjoint sum $L = F + I$, where F is a principal filter and I a principal ideal of $\mathcal{L}$. Thus, $I = \downarrow L_1$ and $F = \uparrow L_2$ for some elements L_1 and L_2. It can be shown that L_1 is $\bigsqcap$-prime and L_2 is $\bigsqcup$-prime. L_2 is called the *splitting companion* of L_1 and is denoted by $\mathcal{L}/L_1$. Every $\bigsqcap$-prime logic has a unique splitting companion. Moreover, every $\bigsqcup$-prime logic is the splitting companion of some $\bigsqcap$-prime logic. This induces an order preserving bijection between $\bigsqcap$-prime and $\bigsqcup$-prime logics. In a continuous lattice this is automatically also an order preserving bijection between the $\bigsqcap$-irredicible and the $\bigsqcap$-irreducible elements.

If a logic is $\bigsqcap$-irreducible it is the theory of a subdirectly irreducible algebra; if L is in addition complete, it is the theory of a one-generated frame. (Recall that a frame $\langle W, R \rangle$ is one-generated iff there is a single point x such that $x \, R^* \, y$ for all $y \in W$, where R^* is the reflexive transitive closure of R.) Furthermore, [1] has established that L is $\bigsqcap$-prime in Ext K iff it is the logic of a one-generated finite, cycle-free frame. Thus, not every one-generated frame determines a splitting logic. This shows that Ext K is not continuous: for the logic of a one-generated finite frame is $\bigsqcap$-irreducible. However, K is the intersection of all its splitting logics. This is because if a formula has a model, it has a model on a finite cycle-free frame, by unravelling.

From this it follows that Ext K can have no atoms. For these must be $\bigsqcup$-irreducible, hence $\bigsqcup$-prime. And so they have the form Ext K/L' for some L'. L' must be a $\bigsqcap$-prime logic, and a minimal one. But it is the logic of a finite frame, and so there is a $\bigsqcap$-prime $L'' < L'$. A similar argument applies to Ext $\mathsf{K4}$, Ext $\mathsf{S4}$ and many other lattices. This explains why we focus here on downgoing chains rather than upgoing chains.

3 WPOs and their height

Take a poset $\mathcal{P} := \langle P, \leqslant \rangle$ that has no infinite descending chains. $\mathcal{P}$ is called a *well-partial order* (WPO) if it has no infinite antichains. A poset is a WPO iff every linear order extending it is a well-order. Denote by $o(\mathcal{P})$ the supremum of all well-orders on P that extend $\leqslant$. Further, by a result of [3], in a WPO, the supremum is actually realised, that is, there is an actual chain of order type $o(\mathcal{P})$ extending $\leqslant$.

A set $A \subseteq P$ is a *lower set* or *ideal* of $\mathcal{P}$ if for all $x \in A$ and $y \leqslant x$ also $y \in A$. The ideals form a distributive lattice. It is known that $\langle P, \leqslant\rangle$ is a WPO iff $\langle \mathcal{I}(\mathcal{P}), \subseteq\rangle$ is well-founded. And in that case, the height of the set $\mathcal{P}$ in this space of ideals is nothing but $o(\mathcal{P})$ ([7]).

We shall use this theory to establish some bounds on chains in lattices of modal logics. In a locale $\mathcal{L}$ (which includes all lattices of the form Ext L for some modal logic L), every logic is the intersection of $\bigsqcap$-irreducible elements. So, given a logic L, let Irr_L be the set of $\bigsqcap$-irreducible logics $\geqslant L$. Irr_L forms a poset. However, note that we are looking here at the reverse inclusion of logics. Thus, to apply the theory of WPOs notice that $L \leqslant L'$ in the notation of WPOs is the same as $L \supseteq L'$. In most cases of interest this order has no infinite descending chains, which translates into an absence of ascending chains of irreducible logics. However, at the end of this paper we shall exhibit a logic with a linear lattice of extensions with an ascending chain of irreducible (even splitting) extensions.

Now, for $L' \supseteq L$ let $h(L') \subseteq \mathrm{Irr}_L$ be the set of all $\bigsqcap$-irreducible logics containing L'; by definition, this set does not depend on the particular base logic L. h is a map from Ext L into $\wp(\mathrm{Irr}_L)$. Its image are lower closed sets in the poset order; equivantly, they are upper closed in the lattice or containment order. However, not all lower closed sets are of the form $h(L')$. This is the case only if Ext L is continuous. (See Chapter 7 of [6] for background. I shall make this paper self-contained, not assuming the heavy structure theory of locales.) In that case, a downgoing chain of logics $\langle L_\lambda : \lambda < \mu\rangle$ thanslates into a well-ordered sequence $\langle h(L_\lambda) : \lambda < \mu\rangle$ of closed sets of Irr_L (ordered by set inclusion). Furthermore, $h(L_{\lambda+1}) - h(L_\lambda) = \{M_\lambda\}$ for a single $\bigsqcap$-irreducible logic M_λ as well as

$$h(L_\lambda) = \bigcup_{\mu<\lambda} h(L_\mu) \tag{3}$$

for a limit ordinal λ. Put $L_\kappa \leqslant L_\mu$ iff $\kappa \leqslant \mu$. This order is clearly compatible with the poset order. Moreover, as all elements of Irr_L are $\bigsqcap$-prime, every element is of the form M_λ for some λ. Thus, a downgoing chain of logics translates into a well-ordering of Irr_L that is consistent with the partial ordering on Irr_L inherited from the lattice. This explains the connection with the theory of [7]. The depth of L is then nothing but $o(\mathrm{Irr}_L)$.

One can establish more than that, however. It is not always necessary to assume that every $\bigsqcap$-irreducible logic is also $\bigsqcap$-prime. However, if that is so, not every linear order on the irreducibles defines a downgoing chain of logics. We can conclude the following general fact.

Theorem 3.1 *Let L be a logic and $\langle \mathrm{Irr}_L, \leqslant\rangle$ be the poset of its $\bigsqcap$-irreducible logics. Then if $\langle \mathrm{Irr}_L, \leqslant\rangle$ is a WPO, the depth of L is $\leqslant o(\langle \mathrm{Irr}_L, \leqslant\rangle)$. Moreover, if every element of Irr_L is also $\bigsqcap$-prime, equality holds.*

Proof. The second claim has been proved already. So we concentrate on the latter. First of all, L has a depth, and so there is a downgoing chain $\langle L_\alpha : \alpha < \kappa\rangle$. This chain translates into an increasing sequence of subsets of Irr_L. As before, $h(L_{\lambda+1}) = h(L_\lambda) \cup \{M_\lambda\}$ for some $\bigsqcap$-irreducible M_λ. However, it is not necessarily the case that

$$h(L_\lambda) = \bigcup_{\mu<\lambda} h(L_\mu) \tag{4}$$

For if L' is $\bigsqcap$-irreducible but not $\bigsqcap$-prime it may happen that $L_\mu \not\leqslant L'$ for all L_μ with $\mu < \lambda$ but $L_\lambda \leqslant L'$. The increasing sequence does therefore not define a linear well-order on the Irr_L; it can be extended to such a sequence. This latter sequence has length $\leqslant o(\langle \mathrm{Irr}_L, \leqslant\rangle)$.

A particular example may suffice to establish the point of divergence. Take the locale $G = 1 + \omega^{op} \times \omega^{op}$, consisting of a bottom element, denoted $\perp$, and all pairs (i, j) of natural numbers. ($\omega^{op} := \langle \omega, \geqslant\rangle$.) Moreover, $(i, j) \leqslant (i', j')$ iff $i \geqslant i'$ and $j \geqslant j'$. Then the $\bigsqcap$-irreducibles are the elements of the form $(i, 0)$ or $(0, j)$. The order type of the space of $\bigsqcap$-irreducibles is $\omega + \omega$. It is worth explaining why. Let $\alpha \# \beta$ denote the Hessenberg-sum of α and β. This is defined on the basis of the so-called Cantor normal form of ordinals. Every ordinal has a representation as a finite sum $\sum_{i<m} \omega^{\gamma_i}$ where the γ_i form a finite (not necessarily strictly) descending sequence. If $\alpha = \sum_{i<m} \omega^{\zeta_i}$ and $\beta = \sum_{j<m} \omega^{\theta_j}$ are in Cantor normal form with descending sequences ζ_i and θ_j, form the sequence η_k, $k < m + n$, by interleaving the ζ_i and θ_j to form a descending sequence again. Then $\alpha \# \beta := \sum_{k<m+n} \omega^{\eta_k}$. Then a result of [3] states that $o(\mathcal{P} + \mathcal{Q}) = o(\mathcal{P}) \# o(\mathcal{Q})$, $\mathcal{P} + \mathcal{Q}$ being the disjoint sum of the posets $\mathcal{P}$ and $\mathcal{Q}$. In the case at hand, $o(\omega + \omega) = \omega \# \omega = \omega + \omega$.

The depth of $\perp$ however is only ω. For consider a downgoing chain of elements of order type ω in G. It consists in a sequence of elements (i_α, j_α) where $i_\alpha \leqslant i_{\alpha'}$ and $j_\alpha \leqslant j_{\alpha'}$ for all $\alpha < \alpha'$. If both the sequence of i_α, $\alpha < \omega$, and j_α, $\alpha < \omega$, are unbounded, the intersection $\bigsqcap_{\alpha<\omega} (i_\alpha, j_\alpha) = \perp$. However, if either sequence is bounded while the other is unbounded, the intersection also equals $\perp$. Hence, all downgoing chains of length ω end in $\perp$. Notice that in this particular locale there are no $\bigsqcap$-prime elements. This situation is not so uncommon in tense logic, see Section 7.9 of [6].

A downgoing chain of logics must be countable because the language is countable. Now, given a logic has a depth at all, either a largest ordinal chain exists or it does not. In the second case the upper limit of these ordinals is ω_1.

Proposition 3.2 *If a modal logic has a depth, the depth is at most ω_1.*

We start with some easy examples. A logic is *pretabular* if it is not tabular but all of its proper extensions are. Tabular logics have finite codimension, but the converse need not hold (take the logic of the so-called veiled recession frame).

Proposition 3.3 *Pretabular logics have depth $\leqslant \omega$.*

Proof. Let L be pretabular. Consider a downgoing chain $\langle L_m : m < \alpha\rangle$ of logics. This chain may be finite. If it is infinite, the logic L_ω does not have finite depth, hence is not tabular. So, L_ω is pretabular and therefore $L = L_\omega$ and $\alpha = \omega$.

In particular, $\mathsf{S5}$ and $\mathsf{Grz.3}$ have depth ω.

Consider next the logic $\mathsf{K.Alt}_1 = \mathsf{K} \oplus \Diamond p \to \Box p$. This is the least normal extension of K containing $\Diamond p \to \Box p$. Every extension of this logic has the finite model property and is finitely axiomatisable ([8]). The generated finite frames are either of the form $\mathfrak{Ch}_n := \langle\{0, 1, \cdots, n-1\}, I_n\rangle$, $i\, I_n\, j$ iff $j = i + 1$, or of the form $\mathfrak{Ch}^\bullet_n := \langle\{0, 1, \cdots, n-1\}, I_n \cup \{\langle n-1, n-1\rangle\}\rangle$. In the lattice Ext $\mathsf{K.Alt}_1$ only the logics $\mathsf{Th}\,\mathfrak{Ch}_n$ are $\bigsqcap$-prime. (As usual, $\mathsf{Th}, \mathfrak{F}$ denotes the logic of the frame $\mathfrak{F}$.) Moreover, their intersection is $\mathsf{K.Alt}_1$. This shows that there is a downgoing chain of ordinal type ω ending at $\mathsf{K.Alt}_1$. However, the ordinal depth of that logic is larger still. The logics $\mathsf{Th}\,\mathfrak{Ch}^\bullet_n$ form a downgoing chain of type ω whose intersection is $\mathsf{K.Alt}_1.\mathsf{D}$. The logics $\mathsf{K.Alt}_1.\mathsf{D} \sqcap \mathsf{Th}\,\mathfrak{Ch}_n$ form a downgoing chain of order type ω from $\mathsf{K.Alt}_1.\mathsf{D}$ ending at $\mathsf{K.Alt}_1$. Concatenating these sequences yields a chain of type $\omega + \omega$. Since we have used up all $\bigsqcap$-irreducible elements, this cannot be improved upon.

Theorem 3.4 *The logic* $\mathsf{K.Alt}_1$ *has depth* $\omega + \omega$.

The logic $\mathsf{K45}$ is another logic with depth $\omega + \omega$. Its space of irreducible elements is different, though. It consists of (i) clusters of size n or (ii) clusters of size n preceded by an irreflexive point. Frames of the type (ii) contain frames of type (i), unlike in the previous case. Nevertheless, there is a chain of type $\omega+\omega$ which first passes through $\mathsf{S5}$, by picking up first the clusters with increasing n and then the frames of type (ii) with increasing n.

It is essential to consider maximal sequences, otherwise the results are trivial. For example, notice that there is another chain in Ext $\mathsf{K.Alt}_1$, formed by the logics $\mathsf{Th}\,\mathfrak{Ch}_n$, which has order type ω, which starts at the top and goes down to $\mathsf{K.Alt}_1$. Thus not all downgoing chains have the same order type, quite unlike the case of logics of finite depth. Moreover, a chain of type ω almost always exists.

Theorem 3.5 *Let L be a logic without finite depth which has finite model property. Then there exists a downgoing chain of order type ω starting at* $\mathsf{K} \oplus \bot$ *whose limit is L. In particular, L has a depth.*

Proof. Let $\{\mathfrak{F}_n : n \in \omega\}$ be an enumeration of the finite one-generated frames for L. Then form the following sequence $S := \langle L_m : m \in \omega + 1\rangle$:

$$\begin{aligned} L_0 &:= \mathsf{K} \oplus \bot \\ L_n &:= \textstyle\bigcap_{m<n} \mathsf{Th}\,\mathfrak{F}_m \\ L_\omega &:= L \end{aligned} \tag{5}$$

This is a descending sequence of logics, that is $L_{n+1} \subseteq L_n$. Since $\mathsf{Th}\,\mathfrak{F}_n$ has finite codimension, $L_{n+1} = L_n \sqcap \mathsf{Th}\,\mathfrak{F}_n$ has finite codimension under L_n. If $L_{n+1} = L_n$, we drop L_{n+1}. We obtain a strictly descending sequence of logics of type $\leqslant \omega$ where each member has finite dimension over the next. To make this a chain, we fill in some finite number of logics at each step. This chain cannot be finite. Hence it is of order type ω. The intersection of its members is L. Hence, there is a descending chain to L of order type $\omega + 1$, making L of depth ω.

A familiar lattice theoretic argument has been used in this proof. Consider elements x, y, z in a distributive lattice such that $y = x \sqcup z$. There are maps $\psi : [z, y] \to [z \sqcap x, x] : u \mapsto u \sqcap x$ and $\chi : [z \sqcap x, x] \to [z, y] : v \mapsto v \sqcup z$. These maps are order preserving; moreover, they are inverses of each other. Assume $z \leqslant u \leqslant y$. Then

$$\chi(\psi(u)) = (u \sqcap x) \sqcup z = (u \sqcup z) \sqcap (x \sqcup z) = u \sqcap y = u \tag{6}$$

Now assume $x \sqcap z \leqslant v \leqslant x$:

$$\psi(\chi(v)) = (v \sqcup z) \sqcap x = (v \sqcap x) \sqcup (z \sqcap x) = v \sqcup (x \sqcap x) = v \tag{7}$$

It follows that the two intervals are isomorphic. A special case worth noting is the case of a cocover. u is a cocover of v iff the interval $[u, v]$ contains exactly two elements. Hence we have the following

Lemma 3.6 *Assume that x, y, z are elements in a distributive lattice such that $x \sqcup z = y$. Then z is a cocover of y iff $x \sqcap z$ is a cocover of x.*

We call the pair (y, z) a *prime quotient*, and write $y \succ z$, to say that z is a cocover of y. One says that intersecting this quotient with x *projects* it onto a prime quotient $x = x \sqcap z \succ x \sqcap y$. In general, if $x \prec y$ then either $x \sqcap z = y \sqcap z$ or $x \sqcap z \prec y \sqcap z$. As a consequence, if y has finite dimension over x then $y \sqcap z$ has finite dimension over $x \sqcap z$ as well. We shall use this type of argument quite frequently.

Notice even though in a modular lattice if x has finite dimension over y then every downward chain from x to y has the same length, this does not carry over to infinite dimensions even if the lattice is continuous (as is the case with Ext K45). The result above suggests that dimension is a cardinal invariant (thus asking about the cardinality of the chain), while depth is construed here as an ordinal number. The price to pay is to define it as the supremum of order types, since they are not unique.

Consider next the logic S4.3.1. Its frames can be represented by sequences $\gamma = \langle c_i : i < m \rangle$ of nonzero numbers, where it is understood that c_i is the size of the cluster of depth $i + 1$. The final cluster (which we define to be of depth 0) has size 1, and is *not* represented in the sequences, which may therefore be empty (in that case $m = 0$). Given a sequence, the frame is constructed over the set $\{(\delta, i) : \delta = i = 0 \textit{ or } \delta - 1 < m \textit{ and } i < c_{\delta-1}\}$, and $(\delta, i) \, R \, (\delta', i')$ iff $\delta \geqslant \delta'$. Now, γ can also be understood as a finite word over $\omega - \{0\}$. Given another such word, $\eta = \langle d_i : i < n \rangle$, the frame of γ is a p-morphic image of η iff there is a strictly ascending map $v : \{0, \cdots, m-1\} \to \{0, \cdots, n-1\}$ such that $d_{v(i)} \geqslant c_i$ for all $i < m$. Also, if γ is a generated subframe of η then η is contractible to γ. Now, $\mathsf{Th}\,\gamma \supseteq \mathsf{Th}\,\eta$ iff γ is a p-morphic image of a generated subframe of η iff γ is a p-morphic image of η iff $\gamma \leqslant \eta$, where $\leqslant$ is the Higman ordering on $(\omega - \{0\})^*$, which in turn is isomorphic to the Higman ordering on ω^*. Here, P^* for a set P denotes the set of finite sequences of elements of U. (The asterisk is thus the so-called Kleene-star.) Given a WPO P, the Higman ordering on P^* is as follows. If $x = x_0 x_1 \cdots x_{m-1}$ for certain $x_i \in P$, then $x \leqslant z$ iff $z = z_0 z_1 \cdots z_{n-1}$, $z_i \in P$, and there is a strictly ascending map $v : \{0, \cdots, m-1\} \to \{0, \cdots, n-1\}$ such that $x_i \leqslant z_{v(i)}$ for all $i < m$.

Lemma 3.7 *Let γ and η represent two finite S4.3.1-frames. Then η is contractible to γ iff $\gamma \leqslant \eta$.*

Proof. Basically, it is known, e. g. [2], that for S4.3-frames $\mathfrak{F}$ and $\mathfrak{G}$, $\mathfrak{F}$ is contractible to $\mathfrak{G}$ iff $\mathfrak{G}$ is a cofinal subframe of $\mathfrak{F}$. Final clusters are of size 1, so the final cluster of $\mathfrak{G}$ is always embeddable in the final cluster of $\mathfrak{F}$. The remainder is the condition that $\gamma \leqslant \eta$, where γ represents $\mathfrak{G}$ and η represents $\mathfrak{F}$.

It turns out that $o(\omega^*) = \omega^{\omega^\omega}$ (Theorem 16 of [7], based on results of [3]).

Theorem 3.8 *The depth of S4.3.1 is ω^{ω^ω}.*

We can extend this result to S4.3. A frame for S4.3 can be represented as a pair (γ, p), where $p > 0$ represents the size of the final cluster, and γ is a sequence representing the sizes of nonfinal clusters as above. Now (γ, p) is subreducible to (γ', p') iff $p \geqslant p'$ and $\gamma \leqslant \gamma'$. Thus, we have a WPO of order type $\omega^* \times \omega$. The following is an application of Theorem 3.5 in [3] stating that the order type of a product of WPOs is the so-called Hessenberg product of the order types of the individual WPOs. Namely, if $\alpha = \sum_{i<m} \omega^{\zeta_i}$ and $\beta = \sum_{j<n} \omega^{\theta_j}$ are in Cantor normal form, put $\alpha \circledast \beta = \sum_{i<mn} \omega^{\eta_i}$, where $\langle \eta_i : i \leqslant mn \rangle$ is a suitable rearrangement of the sequence $\langle \zeta_i \# \theta_j : i \leqslant m, j \leqslant n \rangle$ so as to make the sequence nonincreasing. Applying this to the case at hand, we get $\omega^{\omega^\omega} \circledast \omega = \omega^{\omega^\omega \# 1} = \omega^{\omega^\omega + 1}$.

Theorem 3.9 *The depth of S4.3 is $\omega^{\omega^\omega + 1}$.*

4 Logics with Uncountable Depth

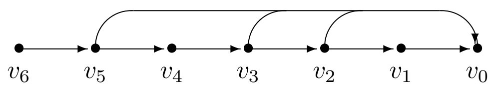
Let us now consider the case when the space of irreducibles is not a WPO. This may essentially have two reasons: the poset of irreducible logics possesses infinite upgoing chains, or it has an infinite antichain. We simplify the matter by looking at splitting elements. We first establish a result that deals with the case where the set of $\prod$-prime logics contains an infinite antichain.

Let $\mathfrak{D} := \langle \{v_0\}, \varnothing \rangle$, $\mathfrak{C}_\varepsilon := \langle \{v_1, v_0\}, \{\langle v_1, v_0 \rangle\} \rangle$. Now define frames $\mathfrak{C}_{\boldsymbol{x}} = \langle W_{\boldsymbol{x}}, R_{\boldsymbol{x}} \rangle$, $\boldsymbol{x} \in \{0,1\}^*$, inductively as follows. Assume that $\boldsymbol{x}$ has length n.

$$\begin{aligned} \mathfrak{C}_{\boldsymbol{x}0} &:= \{W_{\boldsymbol{x}} \cup \{v_{n+2}\}, R_{\boldsymbol{x}} \cup \{\langle v_{n+2}, v_{n+1} \rangle\}\rangle \\ \mathfrak{C}_{\boldsymbol{x}1} &:= \{W_{\boldsymbol{x}} \cup \{v_{n+2}\}, R_{\boldsymbol{x}} \cup \{\langle v_{n+2}, v_{n+1} \rangle, \langle v_{n+2}, v_0 \rangle\}\rangle \end{aligned} \tag{8}$$

See Figure 1 for an example.

Fig. 1. The frame $\mathfrak{C}_{11010}$



Proposition 4.1 *For all $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^*$ the following holds.*

(i) $\mathsf{Th}\,\mathfrak{C}_{\boldsymbol{x}} \subseteq \mathsf{Th}\,\mathfrak{D}$.

(ii) $\mathsf{Th}\,\mathfrak{C}_{\boldsymbol{x}} \subseteq \mathsf{Th}\,\mathfrak{C}_{\boldsymbol{y}}$ *iff $\boldsymbol{y}$ is a prefix of $\boldsymbol{x}$.*

(iii) $\mathsf{Th}\,\mathfrak{C}_{\boldsymbol{x}}$ *has codimension $n+2$, where n is the length of $\boldsymbol{x}$.*

Proof. The first claim is obvious. For the second notice that the frames are noncontractible, one-generated and finite. Hence $\mathsf{Th}\,\mathfrak{C}_{\boldsymbol{x}} \subseteq \mathsf{Th}\,\mathfrak{C}_{\boldsymbol{y}}$ if and only if $\mathfrak{C}_{\boldsymbol{y}}$ is a generated subframe of $\mathfrak{C}_{\boldsymbol{x}}$, which by notation is the case if and only if $\boldsymbol{y}$ is a prefix of $\boldsymbol{x}$.

The logic L_P of these frames can be axiomatized. It is an extension of $\mathsf{K.Alt}_2$, where $\mathsf{Alt}_2 = \bigwedge_{i<3} \Diamond p_i \to \bigvee_{i<j<3} \Diamond(p_i \wedge p_j)$. The additional axiom characterising these frames is

$$\Diamond(p \wedge \Diamond\top) \to \Box(\neg p \to \Box\bot) \tag{9}$$

Indeed, the axiom is first-order stating that for every v such that there are w, w' and w'' with $v\,R\,w$, $v\,R\,w'$, $w\,R\,w''$ and $w \neq w'$ the world w' has no successor.

Let us note that from the results of [1] it follows that

Proposition 4.2 (Blok) $\mathsf{Th}\,\mathfrak{D}$*,* $\mathsf{Th}\,\mathfrak{C}_{\boldsymbol{x}}$ *are $\sqcap$-prime or all $\boldsymbol{x} \in \{0,1\}^*$.*

Define $V \subseteq \{0,1\}^*$. Put $L(\varnothing) := \mathsf{Th}\,\mathfrak{D}$, and for $V \neq \varnothing$ put $L(V) := \bigsqcap_{\boldsymbol{x}\in V} \mathsf{Th}\,\mathfrak{C}_{\boldsymbol{x}}$.

Proposition 4.3 *The following holds for all $V \subseteq \{0,1\}^*$.*

(i) $\mathfrak{C}_{\boldsymbol{y}}$ *is a frame for $L(V)$ iff there is $\boldsymbol{x} \in V$ such that $\boldsymbol{y}$ is a prefix of $\boldsymbol{x}$.*

(ii) *If in addition V is prefix closed then $L(V) \subseteq \mathsf{Th}\,\mathfrak{C}_{\boldsymbol{y}}$ iff $\boldsymbol{y} \in V$.*

Proof. (i) The logic $\mathsf{Th}\,\mathfrak{C}_{\boldsymbol{y}}$ is $\sqcap$-prime. Hence, if $L(V) \subseteq \mathsf{Th}\,\mathfrak{C}_{\boldsymbol{y}}$ there is an $\boldsymbol{x} \in V$ such that $\mathsf{Th}\,\mathfrak{C}_{\boldsymbol{x}} \subseteq \mathsf{Th}\,\mathfrak{C}_{\boldsymbol{y}}$, and conversely. The latter is the case if and only if $\boldsymbol{y}$ is a prefix of $\boldsymbol{x}$, by Proposition 4.1(ii). (ii) If $L(V) \subseteq \mathsf{Th}\,\mathfrak{C}_{\boldsymbol{y}}$ then by (i), $\boldsymbol{y}$ is a prefix of some $\boldsymbol{x} \in V$. Since V is prefix closed, $\boldsymbol{y} \in V$. The converse is clear.

Evidently, if $L = L(V)$, then for the prefix closure V° of V, $L = L(V^\circ)$. So, without loss of generality we may assume a representation of $L(V)$ with a prefix closed set V.

Proposition 4.4 *Let V, W be prefix closed.*

(i) $L(V) \subseteq L(W)$ *iff* $V \supseteq W$.

(ii) $L(W) \prec L(V)$ *iff* $W = V \cup \{\boldsymbol{x}i\}$ *where* $i \in \{0,1\}$, $\boldsymbol{x} \in V$ *but* $\boldsymbol{x}i \notin V$.

Proof. (i) follows from Proposition 4.2. (ii) Suppose that $L(W) \prec L(V)$. Then $W \supsetneq V$ and so there is a $\boldsymbol{y} \in W - V$. Every proper prefix of $\boldsymbol{y}$ must be in V otherwise $L(V \cup \{\boldsymbol{z}\})$ for such a prefix is strictly between $L(W)$ and $L(V)$. So for the largest prefix $\boldsymbol{y}$ of $\boldsymbol{x}$ we have $\boldsymbol{y} \in V$, while $\boldsymbol{x} = \boldsymbol{y}i$ for some $i \in \{0,1\}$. And conversely.

Alternatively, one may use Lemma 5.5 below to prove this proposition. The set $\{0,1\}^*$ forms a complete binary branching tree T under the prefix ordering. Let $U(T)$ be the set of prefix closed subsets of T.

Proposition 4.5 *The map* $V \mapsto L(V)$ *is an order preserving injection from* $\langle U(T), \supseteq\rangle$ *into* Ext L_P.

The mapping is not onto, however. The logic K.D is an extension of L_P that is not of that form.

The tree T has $2^{\aleph_0}$ branches. Each branch defines a logic.

Proposition 4.6 Ext L_P *has an antichain of length* $2^{\aleph_0}$.

Proof. Each branch consists of an infinite linear sequence S of words $\boldsymbol{x}_i$ such that $\boldsymbol{x}_i$ is a prefix of $\boldsymbol{x}_{i+1}$. Obviously, S is prefix closed. If S' is another such set, $L(S) \neq L(S')$. However, if $L(S) \subsetneq L(S')$, then all $\boldsymbol{x}_i$ would be in S', which cannot be the case if S' is a chain different from S. Likewise, $L(S') \subsetneq L(S)$. So the logics form an antichain.

Now let $J := \{0^n 1 : n \in \omega\}$. This is a countably infinite antichain in T. Choose a well-order κ on J, so $J = \{\boldsymbol{x}_\lambda : \lambda < \kappa\}$. κ is countable. Define now the following chain of logics. $L(V_\lambda)$, $\lambda < \omega + \kappa$, with

$$\begin{array}{ll} V_n & := \{0^m : m < n\} \\ V_\omega & := \bigcap_{n\in\omega} V_n \\ V_{\omega+\beta} & := V_\omega \cup \{\boldsymbol{x}_\lambda : \lambda < \beta\} \\ V_\kappa & := L_P \end{array} \tag{10}$$

Proposition 4.7 *(1) $L(V_{\alpha+1})$ is a lower cover of $L(V_\alpha)$ for all $\alpha < \kappa$. (2) If α is a limit ordinal, $L(V_\alpha) = \bigsqcap_{\gamma<\alpha} L(V_\gamma)$.*

Proof. (1) By Proposition 4.4, $L(V_{\alpha+1})$ is a lower cover iff $V_{\alpha+1} - V_\alpha$ contains a single element $\boldsymbol{x}$ such that all its proper prefixes are in V_α. Two cases arise. (Case 1) $\alpha < \omega$. Then $\boldsymbol{x} = 0^\alpha$, and all proper prefixes are indeed in V_α. (Case 2) $\alpha = \omega + \beta$, and $\boldsymbol{x} = 0^n 1$ for some n. All of the proper prefixes are in V_α. (2) is clear.

The following is now evident.

Proposition 4.8 *The $\langle L(V_\alpha) : \alpha < \kappa + 1\rangle$ form a downgoing chain of logics of order type $\kappa + 1$.*

It can easily be shown that for every countable limit ordinal κ there is a downgoing chain from the top of the lattice to L_P.

Proposition 4.9 *L_P has depth ω_1.*

Let us now attack the problem of K. We shall use an abstract argument, which can be applied in a number of cases. It is based on the insight that the intersection of all $\sqcap$-prime logics is K, and that this set contains an infinite antichain.

Proposition 4.10 *Let L be a complete bounded lattice such that the bottom element 0 is the intersection of countably many elements of finite depth. Suppose there exists a countably infinite antichain $\{x_n : n \in \omega\}$ of splitting elements of finite depth. Then the depth of 0 exists and is ω_1.*

Proof. Let S be a countable set such that $\bigsqcap S = 0$. Let $X := \{x_n : n \in \omega\}$ be an antichain of $\sqcap$-prime logics. We can assume that $X \subseteq S$. Let $A := (\uparrow X) - X$, $B := S - \uparrow X$. We choose an enumeration $\langle s_\alpha : \alpha < \eta \rangle$ of S that starts by enumerating the elements from A first, then proceeds with an enumeration of X, and ends in an enumeration of B. The enumerations of A and B can be arbitrary. So, for simplicity we assume A to be of order type ω (if infinite). Let η by an arbitrary countable well-order on X. Then S has order type $\omega + \eta + \omega$, which is at least η. Now define a sequence by

$$y_\alpha := \bigsqcap_{\gamma < \alpha} s_\gamma \tag{11}$$

This is a downgoing sequence of elements with limit $0 = \bigsqcap_{\alpha<\eta} s_\alpha$ (which can be thought of as y_η). Notice also the following. By choice of A and the fact that the elements of X are prime, $X \cap \uparrow y_\omega = \varnothing$. Thus, for $\omega \leqslant \alpha < \omega + \eta$, $y_{\alpha+1} \neq y_\alpha$, so the sequence is actually properly descending at least for $\omega \leqslant \alpha < \omega + \eta$. Now, $y_{\alpha+1}$ is either a lower cover of y_α or it has finite codimension > 1 under y_α.

Hence by inserting finitely many elements between successive elements of the sequence the sequence can be extended to a downgoing chain of order type at least η. η can be choosen arbitrarily. So we can exceed any given countable ordinal. Thus the supremum of these order types is ω_1.

We conclude the following.

Theorem 4.11 *The depth of K is ω_1.*

From [4] we know that the logic Grz_3 of posets of depth at most 3 has an infinite antichain of splitting logics. Thus, any logic contained in Grz_3 with the finite model property has depth ω_1. Similarly for logics contained in G_3 (just take the irreflexive counterpart of Fine's frames).

Theorem 4.12 *$\mathsf{K4}$, $\mathsf{S4}$, G and Grz have depth ω_1.*

5 Logics with and without depth

Consider next the second type of failure, when the poset of irreducibles has infinite ascending chains. In this case not all logics have a depth. Here is an example taken from [6], Page 360. The logic called $\mathsf{G}.\Omega_2$ has an extension lattice isomorphic to $\omega + 2 + \omega^{op}$, by Theorem 7.5.14. ($\omega^{op} := \langle \omega, \geqslant \rangle$. The plus here denotes the ordered sum, making all elements of the first set lower to all elements of the second. It is thus distinct from the independent sum of posets.) Hence, it lacks well-ordered downgoing chains ending in $\mathsf{G}.\Omega_2$. The following summarizes the facts.

Theorem 5.1 Ext $\mathsf{G}.\Omega_2$ *is continuous. It contains an infinite ascending chain of splitting logics.* $\mathsf{G}.\Omega_2$ *has no depth.*

Note that the existence of infinite ascending chains of prime logics is instrumental in establishing lack of depth, as we shall see below.

On the other hand, such logics are hard to construct, as the following series of observations shows.

Lemma 5.2 *If every logic in* Ext L *that properly includes* L *has a cocover,* L *has a depth.*

Proof. For a proof, observe that a descending chain can be constructed inductively as follows. Start with $L_0 := L \oplus \bot$ and let $L_{\kappa+1}$ be a cocover of L_κ in Ext L if $L_\kappa \neq L$; and let $L_\lambda := \bigsqcap_{\mu<\lambda} L_\mu$ for a limit ordinal λ such that $L_\mu \neq L$ for all $\mu < \lambda$.

So on what conditions do cocovers exist? Here are a few cases.

Lemma 5.3 *Let* L' *be finitely axiomatizable over* L*. Then* L' *has a cocover in* Ext L.

Proof. This follows from Tukey's Lemma. Define a property $\mathcal{P}$ of sets of formulae by $\mathcal{P}(\Delta)$ iff $L \oplus \Delta \subsetneq L'$. This is finitely based: it is true of an infinite set if and only if it is true of all its finite subsets. By Tukey's Lemma there is a maximal set Δ^* having $\mathcal{P}$. Then $L \oplus \Delta^*$ is a cocover of L'.

Another example concerns splittings. Notice the following.

Lemma 5.4 *Let* L' *be a splitting logic of* Ext L *with splitting companion* L''*. Then* L'' *has a unique cocover,* $L' \sqcap L''$*. Dually,* L' *has a unique cover,* $L' \sqcup L''$.

Proof. Since Ext L is complete, we can form $L^\circ := \bigsqcup_{M<L''} M$. As L'' is $\bigsqcup$-prime, $L^\circ < L''$. Thus, L° is the cocover of L''. Since L'' is not below L', $L'' \sqcap L' < L''$ and hence $L'' \sqcap L' \leqslant L^\circ$. On the other hand, by definition of a splitting, since $L^\circ < L''$, we must have $L^\circ \leqslant L'$, from which it follows that $L^\circ \leqslant L'' \sqcap L'$. The two logics are thus equal. Hence, $L'' \sqcap L'$ is the (unique) cocover of L''. The other claim is dual.

Lemma 5.5 *Let* L' *be a splitting logic of* L*. If* $M \supseteq (\text{Ext } L)/L'$ *then* M *has a cocover, namely* $M \sqcap L'$.

Proof. By the previous lemma, we have a prime quotient $(\text{Ext } L)/L' \succ L' \sqcap (\text{Ext } L)/L'$. Now let $M \supseteq (\text{Ext } L)/L'$. Consider the quotient $M \succ M \sqcap L'$. Intersecting this quotient with $(\text{Ext } L)/L'$ we get the quotient $M = M \sqcap (\text{Ext } L)/L' \succ (M \sqcap L') \sqcap (\text{Ext } L)/L' = L' \sqcap (M \sqcap (\text{Ext } L)/L') =$

$L' \sqcap (\mathrm{Ext}\ L)/L'$. This is a prime quotient. Hence the original quotient is also prime, by Lemma 3.6.

Proposition 5.6 *Assume that the set of splitting logics of* Ext L *contains no infinite ascending chains. Assume further that L is the intersection of its splitting logics. Then L has a depth.*

Proof. Construct a sequence as follows. L_0 is the inconsistent logic. For a limit ordinal λ, put $L_\lambda = \bigsqcap_{\mu<\lambda} L_\mu$. Assume that $L_\lambda \neq L$. Consider the set $U_\lambda := \{M : M \text{ splits Ext } L, L_\lambda \nsubseteq M\}$. U_λ is not empty, otherwise all splitting logics are already above L_λ, whence $L_\lambda = L$. U_λ contains no infinite ascending chains, hence it has a maximal element L^*. Now put $L_{\lambda+1} := L_\lambda \sqcap L^*$. This is a cocover of L_λ, by Lemma 5.5.

Compare this last result with Theorem 5.1. If the lattice Ext L contains an infinite ascending chain of splitting logics, then the intersection of all splitting logics of Ext L lacks depth.

6 Open Problems

Although K has depth ω_1 we cannot conclude that for any given countable ordinal κ there is a specific logic with depth κ. This remains a problem to be solved.

PROBLEM. Construct for given countable ordinal β a logic of depth β, if such a logic exists.

If Ext L is continuous, then any $\bigsqcap$-irreducible is also $\bigsqcap$-prime. If furthermore Irr_L is a WPO then there is a chain of order type $o(\mathrm{Irr}_L) + 1$. It turns out that this chain is optimal also for any extension of L. That is, if $L' \supseteq L$ then the depth of L' is given as λ where $L' = L_\lambda$ in this well order. This is a consequence of the fact that the space of upper closed sets is order perfect (Theorem 9 in [7]). As a consequence the problem is solved for ordinals below $\omega^{\omega^\omega+1}$.

I close with the following conjecture. Recall that ε_0 is the limit of the sequence $1, \omega, \omega^\omega, \omega^{\omega^\omega}, \cdots$. Equivalently, it is the least fixed point of the ordinal exponentiation function $\beta \mapsto \omega^\beta$. ε_0 is countable.

CONJECTURE. There are no logics of countable depth larger than ε_0.

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