The structure of the lattice of normal extensions of modal logics with cyclic axioms

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Abstract

Irreflexive frames sometimes play a crucial role in the theory of modal logics, although the class of all such frames that consist of only irreflexive points can not be determined by any set of modal formulas. For instance, the modal logic determined by the frame of one irreflexive point is one of the two coatoms of the lattice of all normal modal logics. Another important result is that every rooted cycle-free frame, that consists of irreflexive points only, splits the lattice of all normal modal logics.

In this paper, we consider a family of axioms $\mathbf{Cycl}(n)$ (for $n \geq 0$), which forces frames to be *n-cyclic*. Seeking out the distribution of modal logics of irreflexive frames in the lattice of normal extensions of the modal logic with a cyclic axiom gives us information about the structure of this lattice.

We mainly discuss the case $n = 1$ (the structure of the lattice of normal extensions of $\mathbf{K} \oplus \mathbf{Cycl}(1)$) and the case $n = 2$ (that of normal extensions of $\mathbf{K} \oplus \mathbf{Cycl}(2)$). Finally we discuss the possibility that a similar or a refined argument may bring us information on the structure of the lattice of normal extensions of the logic $\mathbf{K} \oplus \mathbf{Cycl}(n)$ for every $n \geq 1$.

Keywords: irreflexive frame, cyclic axiom, splitting

1 Introduction

In frames for propositional modal logics, each point can be either *reflexive* or *irreflexive* in general. Whereas the class of all reflexive frames can be determined by simple one axiom **T**, the class of all irreflexive frames cannot be characterized by any set of formulas in a usual modal language. However, irreflexive frames sometimes play a crucial role in the theory of modal logics. For instance, the modal logic determined by the frame of one irreflexive point is one of the two coatoms of the lattice of all normal modal logics [8]. Another important example is that every rooted cycle-free frame, that consists of irreflexive points only, splits the lattice of all normal modal logics [1].

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On the other hand, on *cyclicity* of modal algebras, the following result is established.

Theorem 1.1 ([7]) *Let $\mathcal{V}$ be a variety of modal algebras whose signature is finite. The following are equivalent.*

(1) *$\mathcal{V}$ is semisimple.*

(2) *$\mathcal{V}$ is a discriminator variety.*

(3) *$\mathcal{V}$ is weakly-transitive and cyclic.*

This theorem tells us that the cyclicity of modal algebras is as significant as the weak-transitivity of algebras is, although the definition of cyclicity in the above theorem is slightly different from ours. There have been accumulated an enormous amount of works on weakly-transitive modal logics. But as far as we know, there are very few results on cyclic modal logics.

In this paper, we focus on irreflexive frames that also validate the following cyclic axioms. We consider a family of axioms $\mathbf{Cycl}(n)$ (for $n \geq 0$), which forces frames to be *n-cyclic*. Seeking out the distribution of modal logics of irreflexive frames in the lattice of normal extensions of modal logics with a cyclic axiom gives us information about the structure of this lattice.

We mainly discuss the case $n = 1$ (the structure of the lattice of normal extensions of $\mathbf{K} \oplus \mathbf{Cycl}(1)$) and the case $n = 2$ (that of normal extensions of $\mathbf{K} \oplus \mathbf{Cycl}(2)$). Finally we discuss the possibility that a similar or a refined argument may bring us information on the structure of the lattice of normal extensions of the logic $\mathbf{K} \oplus \mathbf{Cycl}(n)$ for every $n \geq 1$.

2 Preliminaries

The propositional modal language is defined in a usual way, where a countably infinite set $Var := \{p_0, p_1, \ldots, p_k, \ldots\}$ of variables is used, and a nullary connective is $\bot$ (falsum), unary connectives are $\neg$ (negation) and $\Box$ (necessity), and a binary connective is $\wedge$ (conjunction). Several others are only abbreviations as, $\top := \neg\bot$, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi := \neg\varphi \vee \psi$, and $\Diamond\varphi := \neg\Box\neg\varphi$. The set of all modal formulas is denoted by Φ, that is also used for the *inconsistent logic*.

A set $\mathbf{L} \subseteq \Phi$ of formulas is called a *normal modal logic*, if it contains: (1) all classical tautologies, and (2) the formula of the form $\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1)$, and is closed under the following rules: (3) Modus Ponens $(\varphi, \varphi \to \psi/\psi)$, (4) Uniform Substitution $(\varphi/\varphi[p_i/\psi])$, and (5) Necessitation $(\varphi/\Box\varphi)$. The smallest normal modal logic on Φ is denoted by $\mathbf{K}$. We call $\mathbf{L}$ simply a *logic* if it is a normal modal logic, since we deal with only propositional normal modal logics.

For a normal modal logic $\mathbf{L}_0$ and a set Γ of formulas, the *smallest normal extension* of $\mathbf{L}_0$ by Γ, or the smallest normal modal logic that contains both $\mathbf{L}_0$ and Γ, is denoted by $\mathbf{L}_0 \oplus \Gamma$. If Γ is a finite set $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of formulas, then the logic $\mathbf{L}_0 \oplus \Gamma$ is simply denoted by $\mathbf{L}_0\varphi_1\varphi_2\cdots\varphi_n$. The class of all *normal extensions* of $\mathbf{L}_0$, that is, the class $\{\mathbf{L} \subseteq \Phi \mid \mathbf{L}_0 \subseteq \mathbf{L} \text{ and } \mathbf{L} \text{ is normal}\}$

is denoted by $\mathrm{NE{\scriptsize XT}}(\mathbf{L}_0)$. This forms a complete lattice, and also satisfies the (finite) distributive law [2].

Two types of mathematical structure are used to interpret modal formulas semantically: one is modal algebras and the other frames.

A *modal algebra* is a structure $\mathfrak{A} := \langle A, \cap, \cup, -, \Box, 1, 0 \rangle$, where $\langle A, \cap, \cup, -, 1, 0 \rangle$ is a Boolean algebra and $\Box$ is a unary modal operator (we use the same symbol as in formulas) that satisfies (a) $\Box 1 = 1$ and (b) $\Box(x \cap y) = \Box x \cap \Box y$.

To interpret a formula φ in a modal algebra $\mathfrak{A}$, we use a *valuation* $v : \Phi \to \mathfrak{A}$ which satisfies the following: (1) $v(\bot) = 0$, (2) $v(\neg\varphi) = -v(\varphi)$, (3) $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$, and $v(\Box\varphi) = \Box(v(\varphi))$. A formula φ is *valid* in an algebra $\mathfrak{A}$, if $v(\varphi) = 1$ holds for any valuation v on $\mathfrak{A}$. For any class $\mathcal{C}$ of modal algebras, the set $\mathbf{L}(\mathcal{C})$ of formulas that are valid in all members of the class $\mathcal{C}$ defines a normal modal logic.

On the other hand, for every modal logic $\mathbf{L}$, a particular class $\mathcal{V} = \mathcal{V}(\mathbf{L})$ of modal algebras corresponds to it. This class $\mathcal{V}$ is called an equationally definable class of modal algebras for $\mathbf{L}$, or the *variety* for $\mathbf{L}$. All subvarieties of $\mathcal{V}(\mathbf{L})$ form a complete lattice, which is dual isomorphic to $\mathrm{NE{\scriptsize XT}}(\mathbf{L})$. Therefore, an investigation of modal logics can also be seen as an investigation of the varieties of modal algebras which correspond to the modal logics considered.

The smallest variety $\mathcal{V}$ that contains a class $\mathcal{C}$ of algebras can be *generated* by: $\mathcal{V} = HSP(\mathcal{C})$, where H, S, P are the following class operators of algebras of a same type. $H(\mathcal{C}) := \{\mathfrak{B} \mid \mathfrak{B} \text{ is a homomorphic image of some } \mathfrak{A} \in \mathcal{C}\}$, $S(\mathcal{C}) := \{\mathfrak{B} \mid \mathfrak{B} \text{ is a subalgebra of some } \mathfrak{A} \in \mathcal{C}\}$, and $P(\mathcal{C}) := \{\mathfrak{B} \mid \mathfrak{B} \text{ is a direct product of some members } \{\mathfrak{A}_i\}_{i \in I} \subseteq \mathcal{C}\}$.

Another way of generating the smallest variety $\mathcal{V}$ which contains a class $\mathcal{C}$ of algebras is: $\mathcal{V} = HP_S(\mathcal{C}_{s.i.})$, where $P_S(\mathcal{D}) := \{\mathfrak{B} \mid \mathfrak{B} \text{ is a subdirect product of some members } \{\mathfrak{A}_i\}_{i \in I} \subseteq \mathcal{D}\}$, and $\mathcal{C}_{s.i.}$ is the class of all *subdirectly irreducible* members in $\mathcal{C}$. Here we do not describe the detail of what the subdirect product is, and what the subdirectly irreducible (s.i. for short) members are. But by this fact, we see that it is the s.i. members in $\mathcal{C}$ that determine the variety $\mathcal{V}$. All s.i. members in $\mathcal{C}$ behave as *building blocks* of the variety $\mathcal{V}$, or dually, the corresponding modal logic. The following algebraic characterization of subdirectly irreducible modal algebras is a key for our analysis.

Theorem 2.1 ([9]) *A non-trivial modal algebra $\mathfrak{A} = \langle A, \cap, \cup, -, \Box, 1, 0 \rangle$ is subdirectly irreducible if and only if there exists an element $d(\neq 1) \in A$, and for any element $x(\neq 1) \in A$, $x \cap \Box x \cap \Box^2 x \cap \cdots \cap \Box^n x \leq d$ holds for some $n \in \omega$.*

The other type of semantics, a (*general*) *frame* is a structure $\mathcal{F} := \langle W, R, P \rangle$, where W is a set of points, R a binary relation on W, and P is a subset of $\mathcal{P}(W)$ which contains W and $\emptyset$, and is closed under $\cap$ (the set-theoretic intersection, $-$ (the set-theoretic complement) and a unary operator $\Box_R$, that is defined as: $\Box_R(X) := \{y \in W \mid \forall x \in W, (xRy \text{ implies } y \in X)\}$ for $X \in \mathcal{P}(W)$.

In order to interpret a formula in a frame, we use a *valuation* $V : \Phi \to P$. For a frame $\mathcal{F} := \langle W, R, P \rangle$, a valuation V, and a point $a \in W$, we define the truth condition of a formula φ at a in a *model* $\langle \mathcal{F}, V\rangle$ ($\langle \mathcal{F}, V\rangle \models_a \varphi$ in symbol) as usual. In particular, $\langle \mathcal{F}, V\rangle \models_a \Box\varphi$ if and only if for any $b \in W$, aRb implies $\langle \mathcal{F}, V\rangle \models_b \varphi$. A formula φ is *valid* in a frame $\mathcal{F}$ ($\mathcal{F} \models \varphi$ in symbol), if $\langle \mathcal{F}, V\rangle \models_a \varphi$ holds for any valuation V on $\mathcal{F}$ and for any point $a \in \mathcal{F}$. For any class $\mathcal{D}$ of frames, the set $\mathbf{L}(\mathcal{D})$ of formulas that are valid in all members in $\mathcal{D}$ defines a normal modal logic. On the other hand, for a normal modal logic $\mathbf{L}$, a frame $\mathcal{F}$ is a *frame for* $\mathbf{L}$ ($\mathcal{F} \models \mathbf{L}$) if $\mathcal{F} \models \varphi$ holds for every $\varphi \in \mathbf{L}$.

There are some formulas (axioms) in normal modal logics that can characterize some classes of frames whose members satisfy a first order condition written in a language with predicate symbols R and $=$. We denote $\mathcal{F} \models\!\!\!\equiv \Xi$ to mean that the frame $\mathcal{F}$ satisfies a condition Ξ. For example, the famous axioms $\mathbf{T} := p \to \Diamond p, \mathbf{B} := p \to \Box\Diamond p$ and $\mathbf{D} := \Diamond\top$ characterize the classes of frames with the following conditions respectively.

Fact 2.2 *For a frame $\mathcal{F} = \langle W, R, P\rangle$,*

(*1*) $\mathcal{F} \models \mathbf{T}$ *if and only if* $\mathcal{F} \models\!\!\!\equiv \forall x(xRx)$.

(*2*) $\mathcal{F} \models \mathbf{B}$ *if and only if* $\mathcal{F} \models\!\!\!\equiv \forall x, y(xRy \text{ implies } yRx)$.

(*3*) $\mathcal{F} \models \mathbf{D}$ *if and only if* $\mathcal{F} \models\!\!\!\equiv \forall x\exists y(xRy)$.

According to the famous Jónsson-Tarski representation [6], for a given frame $\mathcal{F} = \langle W, R, P\rangle$, the modal algebra $\mathcal{F}^*$ which corresponds to the frame $\mathcal{F}$ is constructed as: $\mathcal{F}^* = \langle P, \cap, \cup, -, \Box_R, W, \emptyset\rangle$. The algebra $\mathcal{F}^*$ is indeed a modal algebra, and there is a following correspondence between these two semantics: for any formula $\varphi \in \Phi$, $\mathcal{F} \models \varphi$ if and only if $\mathcal{F}^* \models \varphi$. Conversely, for a given modal algebra $\mathfrak{A}$, we can construct a frame, which is denoted by $\mathfrak{A}_*$ such that both $\mathfrak{A}$ and $\mathfrak{A}_*$ validate the same set of formulas.

We have explained only a minimal set of knowledge and definitions of some technical terms. We will follow notions and nomenclature of modal logics from [4] and those of universal algebras from [3].

3 Irreflexive frames for modal logics with cyclic axioms

3.1 Irreflexive frames of a particular form

Let $\mathcal{F} = \langle W, R, P\rangle$ be a frame. A point $a \in W$ is *irreflexive* if aRa does not hold. We draw an irreflexive point in a frame by a circle ($\circ$). A frame $\mathcal{F}$ is *irreflexive* if every point in $\mathcal{F}$ is irreflexive. In this paper, we employ a family $\{\mathcal{I}_n\}_{n\in\omega}$ and $\mathcal{I}_\infty$ of irreflexive frames of the following form for $n \geq 0$. $\mathcal{I}_n$ is a finite Kripke frame $\langle W, R\rangle$, where $W := \{a_0, a_1, a_2, \cdots, a_n\}$ and $R := \{(a_0, a_k) \mid 1 \leq k \leq n\}$. $\mathcal{I}_\infty := \langle W, R\rangle$, where W is a countably infinite set $\{a_i \mid i \geq 0\}$ and $R := \{(a_0, a_i) \mid i \geq 1\}$. Figures of these frames are below. On the modal logics determined by $\mathcal{I}_n$'s and $\mathcal{I}_\infty$, the following holds.

Proposition 3.1

(1) $\mathbf{L}(\mathcal{I}_0) \supsetneq \mathbf{L}(\mathcal{I}_1) \supsetneq \mathbf{L}(\mathcal{I}_2) \supsetneq \cdots \supsetneq \mathbf{L}(\mathcal{I}_\infty)$.

(2) $\mathbf{L}(\mathcal{I}_\infty) = \bigcap_{i\in\omega} \mathbf{L}(\mathcal{I}_i)$.

Proof. (1) For $k \leq \ell$, there is a p-morphism from $\mathcal{I}_\ell$ to $\mathcal{I}_k$, and so, $\mathbf{L}(\mathcal{I}_\ell) \subseteq \mathbf{L}(\mathcal{I}_k)$. By using the axiom $\mathbf{alt}_n := \Box p_0 \vee \Box(p_0 \to p_1) \vee \cdots \vee \Box((p_0 \wedge p_1 \wedge \cdots \wedge p_{n-1}) \to p_n)$, it is easy to see that $\mathcal{I}_k \models \mathbf{alt}_k$ but that $\mathcal{I}_{k+1} \not\models \mathbf{alt}_k$. Therefore $\mathbf{L}(\mathcal{I}_k) \not\subseteq \mathbf{L}(\mathcal{I}_{k+1})$.
(2) Suppose that $\varphi \notin \mathbf{L}(\mathcal{I}_\infty)$. Then there exists a valuation V and a point a in $\mathcal{I}_\infty$ such that $\langle \mathcal{I}_\infty, V\rangle \not\models_a \varphi$. Since the length of the formula φ is finite, there is $k \in \omega$, a valuation V' on $\mathcal{I}_k$ and a point b in $\mathcal{I}_k$ such that $\langle \mathcal{I}_k, V'\rangle \not\models_b \varphi$. Therefore we have $\mathbf{L}(\mathcal{I}_\infty) \supseteq \bigcap_{i\in\omega} \mathbf{L}(\mathcal{I}_i)$. □

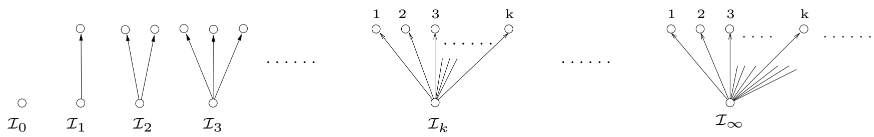

Fig. 1. Frames $\mathcal{I}_k$

3.2 Cyclic axioms and serial axioms

The *cyclic axiom* is defined as: $\mathbf{Cycl}(n) := p \to \Box^n \Diamond p$ for $n \geq 0$. This axiom characterizes the class of frames with the following property.

Fact 3.2 *For a frame $\mathcal{F} = \langle W, R, P\rangle$ and for $n \geq 0$, $\mathcal{F} \models \mathbf{Cycl}(n)$ if and only if $\mathcal{F} \models\!\!\models \forall x_0, x_1, x_2, \ldots, x_n \in W, \big(x_0 R x_1 R x_2 R \cdots R x_n$ implies $x_n R x_0\big)$.*

For $n \geq 0$, a frame $\mathcal{F}$ is *n-cyclic* if $\mathcal{F} \models \mathbf{Cycl}(n)$ holds. Note that this $\mathbf{Cycl}(n)$ is a generalization of the well known axioms $\mathbf{T} := \mathbf{Cycl}(0)$ and $\mathbf{B} := \mathbf{Cycl}(1)$.

On the other hand, the (generalized) *serial* axiom is defined as: $\mathbf{D}_n := \Box^n \Diamond \top$ for $n \geq 0$. This axiom is a generalization of the serial axiom $\mathbf{D}$, and it characterizes the following property of frames.

Fact 3.3 *For a frame $\mathcal{F} = \langle W, R, P\rangle$ and for $n \geq 0$, $\mathcal{F} \models \mathbf{D}_n$ if and only if $\mathcal{F} \models\!\!\models \forall x_0, x_1, \ldots, x_n \in W, \big(x_0 R x_1 R \cdots R x_n$ implies $\exists y \in W (x_n R y)\big)$.*

For $n \geq 0$, a frame $\mathcal{F}$ is *n-serial* if $\mathcal{F} \models \mathbf{D}_n$ holds.

3.3 Levels of points in a frame

Let $\mathcal{F} = \langle W, R, P\rangle$ be a frame. Subsets $W^{(k)}$ for $k = 0, 1, 2, \cdots$ and W^∞ of W is defined as: $W^{(0)} := \{x \in W \mid \mathrm{NOT}\big(\exists y \in W(xRy)\big)\}$, $W^{(n+1)} := \{x \in W \mid \exists y \in W^{(n)}(xRy)\}$, and $W^\infty := \{x \in W \mid \mathrm{NOT}\big(\exists n \in \omega, \exists y \in W^{(0)}(xR^n y)\big)\}$.

In general, these $W^{(k)}$'s and W^∞ are not disjoint. But in a frame for $\mathbf{Cycl}(0)$, $\mathbf{Cycl}(1)$, or $\mathbf{Cycl}(2)$, they are disjoint. And moreover, in a frame for $\mathbf{Cycl}(0)$, $W = W^\infty$, in a frame for $\mathbf{Cycl}(1)$, $W = W^{(0)} \cup W^\infty$, and in a frame for $\mathbf{Cycl}(2)$, $W = W^{(0)} \cup W^{(1)} \cup W^\infty$. This observation is another key fact for our analysis. In these frames, a point $x \in W$ is called a point of *level* n if and only if $x \in W^{(n)}$ for $n = 0, 1$ and x is called a point of *level* ∞ if and only if $x \in W^\infty$.

4 A splitting of NExt(**KCycl**(n)) for $n \geq 1$

Definition 4.1 [Splitting] Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a complete lattice and $a \in L$. Then *a splits* $\mathcal{L}$ if there exists $b \in L$ such that for any $x \in L$, either $x \leq a$ or $b \leq x$, but not both. Such a pair (a, b) is called a *splitting pair* of the lattice $\mathcal{L}$. In this case the element b is a *splitting* of $\mathcal{L}$.

When we think about splittings of a lattice of logics, if a logic $\mathbf{L}(\mathfrak{A})$ (or $\mathbf{L}(\mathcal{F})$) splits the lattice, then we say that the algebra $\mathfrak{A}$ (or the frame $\mathcal{F}$) splits it.

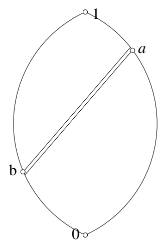

Fig. 2. A splitting of a complete lattice $\mathcal{L}$

A simple calculation for an n-cyclic s.i. modal algebra proves the following lemma first, which leads us to a splitting of the complete lattice NExt(**KCycl**(n)).

Lemma 4.2 *Let $\mathfrak{A}$ be a non-trivial, subdirectly irreducible modal algebra for* **Cycl**(n) $(n \geq 1)$. *Suppose $\Box^{n-1}\Diamond 1 \neq 1$ in $\mathfrak{A}$. Then $\Box^n 0 = 1$ holds in $\mathfrak{A}$.*

Proof. Since $\mathfrak{A}$ is s.i., there is some element $d(\neq 1) \in A$ and for the element $\Box^{n-1}\Diamond 1$, there is a number $m \in \omega$ such that $\Box^{n-1}\Diamond 1 \cap \Box^n\Diamond 1 \cap \cdots \cap \Box^m\Diamond 1 \leq d$ holds. Due to the n-cyclic axiom, we have that $\Box^{n-1}\Diamond 1 \leq d$. Now suppose that $\Box^n 0 \neq 1$ in $\mathfrak{A}$. Then similarly, there is a number $\ell \in \omega$ such that $\Box^n 0 \cap \Box^{n+1} 0 \cap \cdots \cap \Box^\ell 0 \leq d$ holds. Among the conjuncts in the left hand side, $\Box^n 0$ is the smallest, and so, we have $\Box^n 0 \leq d$. By the former, $-d \leq \Diamond^{n-1}\Box 0$, and so, $\Diamond - d \leq \Diamond^n \Box 0 \leq 0$ because of (the dual) of n-cyclic axiom. Therefore we have $\Diamond - d = 0$. Finally we have $-d \leq \Box^n \Diamond - d = \Box^n 0 \leq d$, which implies that $d = 1$. This is a contradiction. Hence we have $\Box^n 0 = 1$. □

Let $\mathcal{C}h_n$ be the irreflexive frame of n point directed chain $(n \geq 1)$. That is, $\mathcal{C}h_n := \langle W, R\rangle$, where $W := \{b_i \mid 0 \leq i \leq n-1\}$ (All b_i's are distinct.) and $R := \{(b_i, b_{i+1}) \mid 0 \leq i \leq n-2\}$. The figure of $\mathcal{C}h_n$ is below.

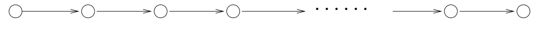

Fig. 3. Frames $\mathcal{C}h_n$

Theorem 4.3 *For any $n \geq 1$, $(\mathbf{KD}_{n-1}\mathbf{Cycl}(n), \mathbf{L}(\mathcal{C}h_n))$ is a splitting pair of the lattice* NExt(**KCycl**(n)).

Proof. Suppose that $\mathbf{KD}_{n-1}\mathbf{Cycl}(n) \not\subseteq \mathbf{L}$ for a logic $\mathbf{L} \in \mathrm{NExt}(\mathbf{KCycl}(n))$. Then, there exists an s.i. algebra $\mathfrak{A}$ for $\mathbf{L}$, such that $\mathfrak{A} \not\models \mathbf{D}_{n-1}(= \Box^{n-1}\Diamond\top)$. This means that $\Box^{n-1}\Diamond 1 \neq 1$ in $\mathfrak{A}$. Therefore, by the previous lemma, $\Box^n 0 = 1$ holds in $\mathfrak{A}$. Here we claim that in $\mathfrak{A}$, $0 < \Box 0 < \Box^2 0 < \cdots < \Box^n 0 = 1$. So, suppose $\Box^k 0 = 1$ for some k $(0 \leq k < n)$. Then, since $0 \leq \Diamond 1$, we have $1 = \Box^k 0 \leq \Box^k \Diamond 1 = 1$, which implies that $\Box^{n-1}\Diamond 1 = 1$. This is a contradiction. Thus $\Box^k 0 \neq 1$ for all k $(0 \leq k < n)$. Suppose that $\Box^k 0 = \Box^{k+1} 0$ for some k $(0 \leq k < n)$. But this leads to a conclusion that $\Box^k 0 = \Box^{k+1} 0 = \Box^{k+2} 0 = \cdots = \Box^n 0 = 1$, which implies that $\Box^k 0 = 1$. This is also a contradiction.

Let $\mathfrak{A}'$ be a subalgebra of $\mathfrak{A}$ generated by the subset $B = \{0, \Box 0, \Box^2 0, \ldots, \Box^{n-1} 0, 1\}$ of A. A map $f : B \to \mathcal{C}h_n{}^*$ is defined as follows: $f(0) := \emptyset, f(\Box 0) := \{b_{n-1}\}, f(\Box^2 0) := \{b_{n-1}, b_{n-2}\}, \ldots, f(\Box^k 0) := \{b_{n-1}, b_{n-2}, \ldots, b_{n-k}\}, \ldots, f(\Box^{n-1} 0) := \{b_{n-1}, b_{n-2}, \ldots, b_1\}$, and $f(1) := \{b_{n-1}, b_{n-2}, \ldots, b_1, b_0\} = W$. Then this f can be extended to a map $F : \mathfrak{A}' \to \mathcal{C}h_n{}^*$ as follows: $F(x) := f(x)$ for $x \in B$ and suppose for any $x, y \in A'$, $F(x)$ and $F(y)$ are already defined. Then we define $F(x \cap y) := F(x) \cap F(y)$, $F(-x) := -F(x)$, and $F(\Box x) := \Box_R(F(x))$. Then, it is easy to see that this F is an embedding, and since the subset $f(B) = \{f(0), f(\Box 0), \ldots, f(\Box^{n-1} 0), f(1)\}$ of the universe of $\mathcal{C}h_n{}^*$ generates the whole $\mathcal{C}h_n{}^*$, and so, F is an isomorphism from $\mathfrak{A}'$ to $\mathcal{C}h_n{}^*$. Therefore $\mathfrak{A}' \cong \mathcal{C}h_n{}^*$, which implies that $\mathcal{C}h_n{}^* \in S(\mathfrak{A})$. Hence we have $\mathbf{L} \subseteq \mathbf{L}(\mathfrak{A}) \subseteq \mathbf{L}(\mathcal{C}h_n{}^*) = \mathbf{L}(\mathcal{C}h_n)$. □

This splitting theorem holds for any $n \geq 1$. We take this theorem as a clue to investigate the structure of the lattice $\mathrm{NExt}(\mathbf{KCycl}(n))$. In what follows, we mainly consider the case $n = 1$ and the case $n = 2$.

5 The structure of NExt(KCycl(1))

Among all connected frames for $\mathbf{Cycl}(1)$, it is only the frame $\mathcal{I}_0$ that contains a point of level 0. By Theorem 4.3, we can see that this frame splits the lattice $\mathrm{NExt}(\mathbf{KCycl}(1))$.

Theorem 5.1 $\big(\mathbf{KD}_0\mathbf{Cycl}(1), \mathbf{L}(\mathcal{I}_0)\big)$ *is a splitting pair of the lattice* $\mathrm{NExt}(\mathbf{KCycl}(1))$. □

Now a question arises: what sort of modal logics are located under the logic $\mathbf{L}(\mathcal{I}_0)$ in $\mathrm{NExt}(\mathbf{KCycl}(1))$? Since $\mathcal{I}_0$ is a frame of only one irreflexive point, it is impossible that this frame is a p-morphic image of some connected frames. It is the case that some suitable generated subframes of $\mathcal{I}_n$ for $n \geq 1$ or $\mathcal{C}h_k$ for $k \geq 1$ are isomorphic to $\mathcal{I}_0$, but they are not frames for $\mathbf{Cycl}(1)$. Our argument in this section will proceed to give an answer to this question.

First of all, the following equality holds.

Proposition 5.2 $\mathbf{KCycl}(1) = \mathbf{KD}_0\mathbf{Cycl}(1) \cap \mathbf{L}(\mathcal{I}_0)$.

Proof. It is obvious that $\mathbf{KCycl}(1) \subseteq \mathbf{KD}_0\mathbf{Cycl}(1) \cap \mathbf{L}(\mathcal{I}_0)$. Conversely suppose $\varphi \notin \mathbf{KCycl}(1)$ for some $\varphi \in \Phi$. Then, there exists a frame $\mathcal{F} = \langle W, R, P \rangle$ for $\mathbf{Cycl}(1)$, a valuation V on $\mathcal{F}$, and a point $a \in W$ such that $\langle \mathcal{F}, V\rangle \not\models_a \varphi$. Now since this $\mathcal{F}$ is 1-cyclic, $W = W^{(0)} \cup W^{\infty}$. If $\mathcal{F}$ is also 0-serial, then $W^{(0)} = \emptyset$ and it is a frame for $\mathbf{Cycl}(1)$ and $\mathbf{D}_0$. Thus we have $\varphi \notin \mathbf{KD}_0\mathbf{Cycl}(1)$. Otherwise, $W^{(0)} \neq \emptyset$. However, since $\mathcal{F}$ is 1-cyclic, any point in $W^{(0)}$ is isolated from other part of the frame. So, if $a \in W^{\infty}$, then the subframe $\mathcal{F}'$ generated by the singleton $\{a\}$ is both 1-cyclic and 0-serial. With the valuation V' which is a restriction of V to $\mathcal{F}'$, we have $\langle \mathcal{F}', V'\rangle \not\models_a \varphi$. This means that $\varphi \notin \mathbf{KD}_0\mathbf{Cycl}(1)$. If $a \in W^{(0)}$, then the subframe $\mathcal{F}''$ generated by the singleton $\{a\}$ is just the frame $\mathcal{I}_0$. With the valuation V'' which is a restriction of V to $\mathcal{F}''$, $\langle \mathcal{F}'', V''\rangle \not\models_a \varphi$ holds. This means that $\varphi \notin \mathbf{L}(\mathcal{I}_0)$. Therefore we have $\varphi \notin \mathbf{KD}_0\mathbf{Cycl}(1) \cap \mathbf{L}(\mathcal{I}_0)$. Hence $\mathbf{KCycl}(1) \supseteq \mathbf{KD}_0\mathbf{Cycl}(1) \cap \mathbf{L}(\mathcal{I}_0)$.□

For normal modal logics $\mathbf{L}_1, \mathbf{L}_2$ such that $\mathbf{L}_1 \subseteq \mathbf{L}_2$, the *interval* between these two logics is denoted by $[\mathbf{L}_1, \mathbf{L}_2]$, that is, $[\mathbf{L}_1, \mathbf{L}_2] := \{\mathbf{L} \in \mathrm{NExt}(\mathbf{K}) \mid \mathbf{L}_1 \subseteq \mathbf{L} \subseteq \mathbf{L}_2\}$.

Define maps σ and τ as follows: $\sigma : \mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(1)) \to [\mathbf{KCycl}(1), \mathbf{L}(\mathcal{I}_0)]$ is defined as: $\sigma(\mathbf{L}) := \mathbf{L} \cap \mathbf{L}(\mathcal{I}_0)$. $\tau : [\mathbf{KCycl}(1), \mathbf{L}(\mathcal{I}_0)] \to \mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(1))$ is defined as: $\tau(\mathbf{M}) := \mathbf{M} \oplus \mathbf{KD}_0\mathbf{Cycl}(1)$. Then on the map σ we can show the following facts.

Lemma 5.3 *σ is a lattice-homomorphism.*

Proof. For logics $\mathbf{L}_1, \mathbf{L}_2 \in \mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(1))$, $\sigma(\mathbf{L}_1 \cap \mathbf{L}_2) = \mathbf{L}_1 \cap \mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_0) = \mathbf{L}_1 \cap \mathbf{L}(\mathcal{I}_0) \cap \mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_0) = \sigma(\mathbf{L}_1) \cap \sigma(\mathbf{L}_2)$. Since the lattice of normal modal logics is distributive, $\sigma(\mathbf{L}_1 \oplus \mathbf{L}_2) = (\mathbf{L}_1 \oplus \mathbf{L}_2) \cap \mathbf{L}(\mathcal{I}_0) = (\mathbf{L}_1 \cap \mathbf{L}(\mathcal{I}_0)) \oplus (\mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_0)) = \sigma(\mathbf{L}_1) \oplus \sigma(\mathbf{L}_2)$. □

Lemma 5.4 *σ is one to one.*

Proof. Suppose $\mathbf{L}_1 \not\subseteq \mathbf{L}_2$ for logics $\mathbf{L}_1, \mathbf{L}_2 \in \mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(1))$. Then there exists a formula φ such that $\varphi \in \mathbf{L}_1$ and $\varphi \notin \mathbf{L}_2$. By the latter, there is a frame $\mathcal{F} = \langle W, R, P\rangle$ for $\mathbf{D}_0, \mathbf{Cycl}(1)$, a valuation V on $\mathcal{F}$ and a point $a \in W$ such that $\langle \mathcal{F}, V\rangle \not\models_a \varphi$. Here, because the frame $\mathcal{F}$ is 0-serial and 1-cyclic, there exists a point $b \in W$ such that $aRbRa$. Since we have $a \not\models \varphi$, $a \not\models \Box^2\varphi$ is also the case. This means that $\langle \mathcal{F}, V\rangle \not\models_a \varphi \vee \Box^2\varphi$, and so, $\varphi \vee \Box^2\varphi \notin \mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_0)$.

On the other hand, since $\Box\bot \to \Box^2\varphi \in \mathbf{K} \subseteq \mathbf{L}(\mathcal{I}_0)$, $\Box^2\varphi \in \mathbf{L}(\mathcal{I}_0)$. Therefore, we have $\varphi \vee \Box^2\varphi \in \mathbf{L}_1 \cap \mathbf{L}(\mathcal{I}_0)$. Thus, $\sigma(\mathbf{L}_1) = \mathbf{L}_1 \cap \mathbf{L}(\mathcal{I}_0) \not\subseteq \mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_0) = \sigma(\mathbf{L}_2)$. □

Lemma 5.5 *σ is onto.*

Proof. Due to the distributivity, and by Proposition 5.2, for any $\mathbf{M} \in [\mathbf{KCycl}(1), \mathbf{L}(\mathcal{I}_0)]$, $\sigma \circ \tau(\mathbf{M}) = (\mathbf{M} \oplus \mathbf{KD}_0\mathbf{Cycl}(1)) \cap \mathbf{L}(\mathcal{I}_0) = (\mathbf{M} \cap \mathbf{L}(\mathcal{I}_0)) \oplus (\mathbf{KD}_0\mathbf{Cycl}(1) \cap \mathbf{L}(\mathcal{I}_0)) = \mathbf{M} \oplus \mathbf{KCycl}(1) = \mathbf{M}$. Hence the map σ is onto. □

So far, we have established the following theorem.

Theorem 5.6 $[\mathbf{KCycl}(1), \mathbf{L}(\mathcal{I}_0)]$ *is isomorphic to* $\mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(1))$.

By Lemma 5.5, we see that for any $\mathbf{M} \in [\mathbf{KCycl}(1), \mathbf{L}(\mathcal{I}_0)]$, there exists a logic $\mathbf{L} \in \mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(1))$ such that $\mathbf{M} = \mathbf{L} \cap \mathbf{L}(\mathcal{I}_0)$. This answers our question in the front of this section.

NEXT(**KCycl**(1)) looks like a *two-story building* as drawn below.

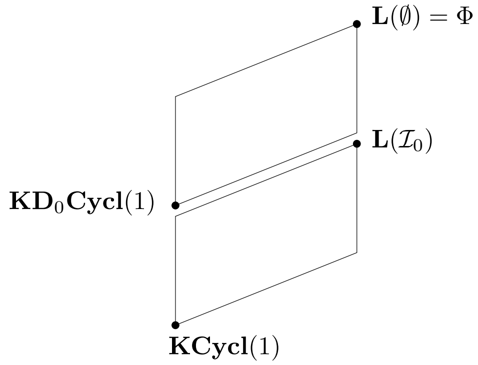

Fig. 4. The structure of NEXT(**KCycl**(1))

6 The structure of NEXT(KCycl(2))

In the case of $n = 2$, the situation is a little different. A frame for the axiom **Cycl**(2) consists of points of level 0, level 1, and level ∞ only. Examples of connected irreflexive frames for this axiom are the family $\{\mathcal{I}_i\}_{i\in\omega}$ and $\mathcal{I}_\infty$, whose logics form an infinite descending chain as shown in Proposition 3.1.

First of all, by Theorem 4.3 we can find one splitting pair.

Theorem 6.1 $\big(\mathbf{KD}_1\mathbf{Cycl}(2), \mathbf{L}(\mathcal{I}_1)\big)$ *is a splitting pair of the lattice* NEXT(**KCycl**(2)).

Note that the frame $\mathcal{I}_0$ is a frame for $\mathbf{D}_1$ and **Cycl**(2), and so, $\mathbf{KD}_1\mathbf{Cycl}(2) \subseteq \mathbf{L}(\mathcal{I}_0)$, whereas all the members in $\{\mathcal{I}_k\}_{k\geq 1}$ and $\mathcal{I}_\infty$ are not 1-serial, and so, $\mathbf{L}(\mathcal{I}_k), \mathbf{L}(\mathcal{I}_\infty) \subseteq \mathbf{L}(\mathcal{I}_1)$. We can find another spitting pair in NEXT(**KCycl**(2)).

Lemma 6.2 *Let* $\mathfrak{A}$ *be a non-trivial s.i. modal algebra for* $\mathbf{D}_1$ *and* **Cycl**(2). *Suppose* $\Diamond 1 \neq 1$ *in* $\mathfrak{A}$. *Then* $\Box 0 = 1$ *in* $\mathfrak{A}$.

Proof. Since $\mathfrak{A}$ is s.i., there is some element $d(\neq 1) \in A$ and for the element $\Diamond 1$, there is a number $m \in \omega$ such that $\Diamond 1 \cap \Box\Diamond 1 \cap \Box^2\Diamond 1 \cap \cdots \cap \Box^m\Diamond 1 \leq d$ holds. Due to the 1-serial axiom and the 2-cyclic axiom, we have that $\Diamond 1 \leq d$. Now suppose that $\Box 0 \neq 1$ in $\mathfrak{A}$. Then similarly, there is a number $\ell \in \omega$ such that $\Box 0 \cap \Box^1 0 \cap \cdots \cap \Box^\ell 0 \leq d$ holds, and so, we have $\Box 0 \leq d$. By these two inequalities, $-d \leq -\Diamond 1 = \Box 0 \leq d$, which leads to a contradiction. Hence we have $\Box 0 = 1$ in $\mathfrak{A}$. □

Theorem 6.3 $\big(\mathbf{KD}_0\mathbf{Cycl}(2), \mathbf{L}(\mathcal{I}_0)\big)$ *is a splitting pair of the lattice* NEXT($\mathbf{KD}_1$**Cycl**(2)).

Proof. Suppose $\mathbf{KD}_0\mathbf{Cycl}(2) \not\subseteq \mathbf{L}$ for some logic $\mathbf{L} \in$ NEXT($\mathbf{KD}_1\mathbf{Cycl}(2)$). Then there exists an s.i. algebra for $\mathbf{L}, \mathbf{D}_1$ and **Cycl**(2) such that $\mathfrak{A} \not\models \mathbf{D}_0$.

This means that $\Diamond 1 \neq 1$ in $\mathfrak{A}$. By the previous lemma, $\Box 0 = 1$ holds in $\mathfrak{A}$. Consider the subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ generated by the set of elements $\{0,1\}$, then it is just the modal algebra $\mathcal{I}_0{}^*$. Therefore we have $\mathcal{I}_0{}^* \in S(\mathfrak{A})$, and so, $\mathbf{L} \subseteq \mathbf{L}(\mathfrak{A}) \subseteq \mathbf{L}(\mathcal{I}_0{}^*) = \mathbf{L}(\mathcal{I}_0)$. $\Box$

By Theorem 6.1 and Theorem 6.3, the following is proved.

Corollary 6.4 $\big(\mathbf{KD_0Cycl}(2), \mathbf{L}(\mathcal{I}_0)\big)$ *is also a splitting pair of the lattice* NEXT($\mathbf{KCycl}(2)$).

Now another question arises: what sort of modal logics are located under the logic $\mathbf{L}(\mathcal{I}_0)$, or under the logic $\mathbf{L}(\mathcal{I}_k)$ for $k \geq 1$ in NEXT($\mathbf{KCycl}(2)$)? Does there exist any intriguing structure as in NEXT($\mathbf{KCycl}(1)$)?

We will compare the top-most part NEXT($\mathbf{KD_0Cycl}(2)$) with the bottom part $[\mathbf{KCycl}(2), \mathbf{L}(\mathcal{I}_\infty)]$ in NEXT($\mathbf{KCycl}(2)$) first.

Proposition 6.5 $\mathbf{KCycl}(2) = \mathbf{KD_0Cycl}(2) \cap \mathbf{L}(\mathcal{I}_\infty)$.

Proof. It is trivial that $\mathbf{KCycl}(2) \subseteq \mathbf{KD_0Cycl}(2) \cap \mathbf{L}(\mathcal{I}_\infty)$. Conversely suppose $\varphi \notin \mathbf{KCycl}(2)$ for some $\varphi \in \Phi$. Then, there exists a frame $\mathcal{F} = \langle W, R, P\rangle$ for $\mathbf{Cycl}(2)$, a valuation V on $\mathcal{F}$, and a point $a \in W$ such that $\langle \mathcal{F}, V\rangle \not\models_a \varphi$. Now since this $\mathcal{F}$ is 2-cyclic, $W = W^{(0)} \cup W^{(1)} \cup W^\infty$ and points in $W^{(0)} \cup W^{(1)}$ are isolated from the rest part in $\mathcal{F}$. So, if $a \in W^{(0)}$, then the subframe generated by the singleton $\{a\}$ is just $\mathcal{I}_0$. Therefore $\varphi \notin \mathbf{L}(\mathcal{I}_0) \supseteq \mathbf{L}(\mathcal{I}_\infty)$. If $a \in W^{(1)}$, the subframe generated by $\{a\}$ is $\mathcal{I}_k$ for some $k \in \omega$ or $\mathcal{I}_\infty$. Therefore $\varphi \notin \mathbf{L}(\mathcal{I}_\infty)$. If $a \in W^\infty$, then the subframe $\mathcal{F}'$ generated by $\{a\}$ contains no element in $W^{(0)} \cup W^{(1)}$, which means that the frame $\mathcal{F}'$ is also a frame for $\mathbf{D}_0$. Therefore $\varphi \notin \mathbf{KD_0Cycl}(2)$. Hence we proved that $\mathbf{KCycl}(2) \supseteq \mathbf{KD_0Cycl}(2) \cap \mathbf{L}(\mathcal{I}_\infty)$. $\Box$

Similarly in the case $n = 1$, we define maps σ_∞ and τ_∞ as follows: σ_∞ : NEXT($\mathbf{KD_0Cycl}(2)$) $\to [\mathbf{KCycl}(2), \mathbf{L}(\mathcal{I}_\infty)]$ is defined as: $\sigma_\infty(\mathbf{L}) := \mathbf{L} \cap \mathbf{L}(\mathcal{I}_\infty)$. $\tau_\infty : [\mathbf{KCycl}(2), \mathbf{L}(\mathcal{I}_\infty)] \to$ NEXT($\mathbf{KD_0Cycl}(2)$) is defined as: $\tau_\infty(\mathbf{M}) := \mathbf{M} \oplus \mathbf{KD_0Cycl}(1)$. Then on the map σ_∞ we can show the following facts also in this case.

Lemma 6.6

(1) σ_∞ *is a lattice-homomorphism.*

(2) σ_∞ *is onto.*

(3) σ_∞ *is one to one.*

Proof. The fact (1) can be obtained immediately by an easy calculation. To show the fact (2), we use Proposition 6.5. To show the fact (3), suppose $\mathbf{L}_1 \not\subseteq \mathbf{L}_2$ for logics $\mathbf{L}_1, \mathbf{L}_2 \in$ NEXT($\mathbf{KD_0Cycl}(2)$). Then there exists a formula φ such that $\varphi \in \mathbf{L}_1$ and $\varphi \notin \mathbf{L}_2$. By the latter, there is a frame $\mathcal{F} = \langle W, R, P\rangle$ for $\mathbf{D}_0, \mathbf{Cycl}(2)$, a valuation V on $\mathcal{F}$ and a point $a \in W$ such that $\langle \mathcal{F}, V\rangle \not\models_a \varphi$. Now the frame $\mathcal{F}$ is 0-serial and 2-cyclic, $a \in W = W^\infty$ and there exist points $b, c \in W$ such that $aRbRcRa$. Since we have $a \not\models \varphi$, $a \not\models \Box^3\varphi$ is also the case.

This means that $\langle \mathcal{F}, V\rangle \not\models_a \varphi \vee \Box^3\varphi$, and so, $\varphi \vee \Box^3\varphi \notin \mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_\infty)$.

On the other hand, since $\Box^2\bot \to \Box^3\varphi \in \mathbf{K} \subseteq \mathbf{L}(\mathcal{I}_\infty)$, $\Box^3\varphi \in \mathbf{L}(\mathcal{I}_\infty)$. Therefore, we have $\varphi \vee \Box^3\varphi \in \mathbf{L}_1 \cap \mathbf{L}(\mathcal{I}_\infty)$. Thus, $\sigma_\infty(\mathbf{L}_1) = \mathbf{L}_1 \cap \mathbf{L}(\mathcal{I}_\infty) \not\subseteq \mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_\infty) = \sigma_\infty(\mathbf{L}_2)$. □

This lemma states that the map σ_∞ is an isomorphism. So far, we have shown the following.

Theorem 6.7 $[\mathbf{KCycl}(2), \mathbf{L}(\mathcal{I}_\infty)]$ *is isomorphic to* $\mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(2))$.

Next, we will compare the top-most part $\mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(2))$ with an interval $[\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_k), \mathbf{L}(\mathcal{I}_k)]$ for $k \geq 0$ in the lattice $\mathrm{NExt}(\mathbf{KCycl}(2))$.

As in the above analysis, we define maps σ_k and τ_k as follows: $\sigma_k : \mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(2)) \to [\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_k), \mathbf{L}(\mathcal{I}_k)]$ is defined as: $\sigma_k(\mathbf{L}) := \mathbf{L} \cap \mathbf{L}(\mathcal{I}_k)$. $\tau_k : [\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_k), \mathbf{L}(\mathcal{I}_k)] \to \mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(2))$ is defined as: $\tau_k(\mathbf{M}) := \mathbf{M} \oplus \mathbf{KD}_0\mathbf{Cycl}(2)$. Then on the map σ_k we can also show the following facts in this case.

Lemma 6.8

(1) σ_k *is a lattice-homomorphism.*

(2) σ_k *is onto.*

(3) σ_k *is one to one.*

Proof. Proofs of the fact (1) and the fact (2) are just similar for the proofs of Lemma 6.6 To show the fact (3), suppose $\mathbf{L}_1 \not\subseteq \mathbf{L}_2$ for logics $\mathbf{L}_1, \mathbf{L}_2 \in \mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(2))$. Then there exists a formula φ such that $\varphi \in \mathbf{L}_1$ and $\varphi \notin \mathbf{L}_2$. By the latter, there is a frame $\mathcal{F} = \langle W, R, P\rangle$ for $\mathbf{D}_0, \mathbf{Cycl}(2)$, a valuation V on $\mathcal{F}$ and a point $a \in W$ such that $\langle \mathcal{F}, V\rangle \not\models_a \varphi$. Now we see that $a \in W = W^\infty$ and there exist points $b, c \in W$ such that $aRbRcRa$. Since we have $a \not\models \varphi$, $a \not\models \Box^3\varphi$ is also the case. This means that $\langle \mathcal{F}, V\rangle \not\models_a \varphi \vee \Box^3\varphi$, and so, $\varphi \vee \Box^3\varphi \notin \mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_k)$.

On the other hand, for the case $k = 0$, $\Box\bot \to \Box^3\varphi \in \mathbf{K} \subseteq \mathbf{L}(\mathcal{I}_0)$ and for the case $k \geq 1$, $\Box^2\bot \to \Box^3\varphi \in \mathbf{K} \subseteq \mathbf{L}(\mathcal{I}_k)$. Therefore for any $k \geq 0$, $\Box^3\varphi \in \mathbf{L}(\mathcal{I}_k)$. Thus we have $\varphi \vee \Box^3\varphi \in \mathbf{L}_1 \cap \mathbf{L}(\mathcal{I}_k)$. Hence, $\sigma_k(\mathbf{L}_1) = \mathbf{L}_1 \cap \mathbf{L}(\mathcal{I}_k) \not\subseteq \mathbf{L}_2 \cap \mathbf{L}(\mathcal{I}_k) = \sigma_k(\mathbf{L}_2)$. □

We have just established the following theorem.

Theorem 6.9 *For any* $k \geq 0$, $[\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_k), \mathbf{L}(\mathcal{I}_k)]$ *is isomorphic to* $\mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(2))$.

Finally we show that there are countably infinite splitting pairs in the lattice $\mathrm{NExt}(\mathbf{KCycl}(2))$ by using these isomorphisms σ_k's. The following fact must be checked.

Proposition 6.10 $\mathbf{KD}_1\mathbf{Cycl}(2) = \mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_0)$.

Proof. It is trivial that $\mathbf{KD}_1\mathbf{Cycl}(2) \subseteq \mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_0)$. Conversely suppose $\varphi \notin \mathbf{KD}_1\mathbf{Cycl}(2)$ for some $\varphi \in \Phi$. Then, there exists a frame $\mathcal{F} = \langle w, R, P\rangle$ for $\mathbf{D}_1, \mathbf{Cycl}(2)$, a valuation V on $\mathcal{F}$, and a point $a \in W$ such that

$\langle \mathcal{F}, V \rangle \not\models_a \varphi$. Now since this $\mathcal{F}$ is 1-serial and 2-cyclic, $W = W^{(0)} \cup W^{\infty}$ and points in $W^{(0)}$ are isolated from the rest part in $\mathcal{F}$. So, if $a \in W^{(0)}$, then the subframe generated by the singleton $\{a\}$ is just $\mathcal{I}_0$. Therefore $\varphi \notin \mathbf{L}(\mathcal{I}_0)$. If $a \in W^{\infty}$, then the subframe $\mathcal{F}'$ generated by $\{a\}$ contains no element in $W^{(0)}$ at all, which means that $\mathcal{F}'$ is also a frame for $\mathbf{D}_0$. Therefore $\varphi \notin \mathbf{KD}_0\mathbf{Cycl}(2)$. Hence we proved that $\mathbf{KD}_1\mathbf{Cycl}(2) \supseteq \mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_0)$. □

Theorem 6.11 *For any $k \geq 1$, the pair $\big(\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_{k-1}), \mathbf{L}(\mathcal{I}_k)\big)$ is a splitting pair in* NExt($\mathbf{KCycl}(2)$).

Proof. By Theorem 6.9 and Proposition 6.10, the interval $[\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_0), \Phi]$ is isomorphic to $[\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_k), \mathbf{L}(\mathcal{I}_{k-1})]$. By Theorem 6.3, $\big(\mathbf{KD}_0\mathbf{Cycl}(2), \mathbf{L}(\mathcal{I}_0)\big)$ is a splitting pair in NExt($\mathbf{KD}_1\mathbf{Cycl}(2)$) $=$ $[\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_0), \Phi]$. By the above isomorphisms, $\mathbf{KD}_0\mathbf{Cycl}(2)$ is mapped to $\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_{k-1})$, and $\mathbf{L}(\mathcal{I}_0)$ to $\mathbf{L}(\mathcal{I}_k)$. Therefore, the pair $\big(\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_{k-1}), \mathbf{L}(\mathcal{I}_k)\big)$ is a splitting pair in the interval $[\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_k), \mathbf{L}(\mathcal{I}_{k-1})]$. Hence this pair is also a splitting pair in NExt($\mathbf{KCycl}(2)$). □

Corollary 6.12 *There exist at least countably infinite splitting pairs in* NExt($\mathbf{KCycl}(2)$).

In this case, NExt($\mathbf{KCycl}(2)$) looks like a *ω-story building* as drawn below.

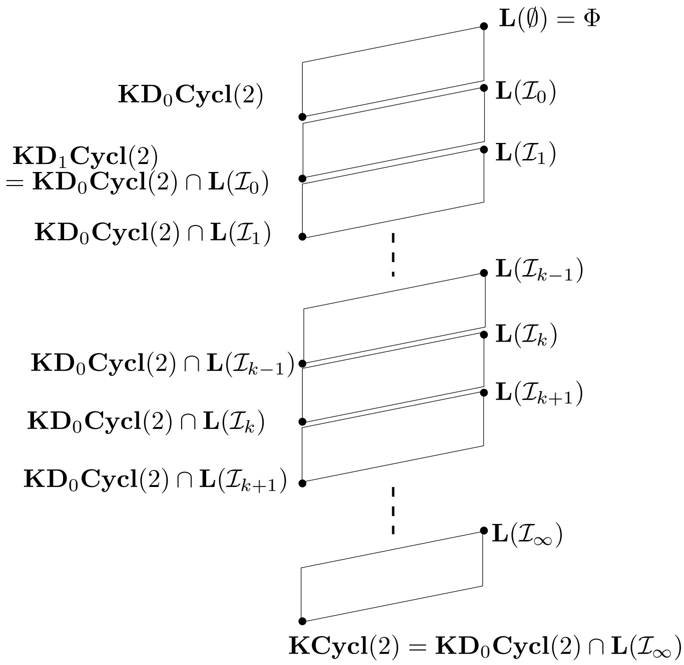

Fig. 5. The structure of NExt($\mathbf{KCycl}(2)$)

7 Outlook

For $n \geq 1$, let $\mathcal{I}rr_n$ be the class of normal modal logics above $\mathbf{KCycl}(n)$ that are determined by a class of *irreflexive* frames. That is, $\mathcal{I}rr_n := \{\mathbf{L}(\mathcal{C}) \in \mathrm{NExt}(\mathbf{KCycl}(n)) \mid \mathcal{C}$ is a class of some irreflexive frames$\}$. Then this $\mathcal{I}rr_n$ forms at least a meet-semilattice. In particular, $\mathcal{I}rr_1$ forms a two element chain ($\Phi = \mathbf{L}(\emptyset)$ and $\mathbf{L}(\mathcal{I}_0)$), and $\mathcal{I}rr_2$ forms an infinite descending chain $(\Phi, \mathbf{L}(\mathcal{I}_0), \mathbf{L}(\mathcal{I}_1), \mathbf{L}(\mathcal{I}_2), \cdots, \mathbf{L}(\mathcal{I}_k), \cdots, \mathbf{L}(\mathcal{I}_\infty))$. In this terminology, the results presented in this paper are summed up in the following way. In both cases there exists a essential lattice structure $\mathcal{B}_n$ at the top of the lattice $\mathrm{NExt}(\mathbf{KCycl}(n))$ and the whole lattice can be expressed in:

$$\mathbf{KCycl}(n) \cong \mathcal{B}_n \times \mathcal{I}rr_n$$

for $n = 1, 2$. Does this beautiful expression also hold for the cases $n \geq 3$? If it is solved in the affirmative, then each lattice $\mathrm{NExt}(\mathbf{KCycl}(n))$ has its own essential part $\mathcal{B}_n$ (it may be at around the top-most region) for any $n \geq 1$, and so, the investigation of such lattice of logics can be concentrated only in this $\mathcal{B}_n$.

In this paper, we discover a splitting pair in $\mathrm{NExt}(\mathbf{KCycl}(1))$, and countably infinite splitting pairs in $\mathrm{NExt}(\mathbf{KCycl}(2))$. Is there any other splitting pair in $\mathrm{NExt}(\mathbf{KCycl}(1))$ or in $\mathrm{NExt}(\mathbf{KCycl}(2))$?

In [5], K. Fine introduced a notion of the degree of Kripke incompleteness of a modal logic as follows: For modal logics $\mathbf{L}_1, \mathbf{L}_2 \in \mathrm{NExt}(\mathbf{L}_0)$, $\mathbf{L}_1$ and $\mathbf{L}_2$ are *Kripke equivalent* ($\mathbf{L}_1 \equiv_K \mathbf{L}_2$ in symbol), if for any Kripke frame $\mathcal{F}$, $\mathcal{F} \models \mathbf{L}_1$ if and only if $\mathcal{F} \models \mathbf{L}_2$. *The degree of Kripke incompleteness* of $\mathbf{L}$ over $\mathbf{L}_0$ ($\delta_{\mathbf{L}_0}(\mathbf{L})$) is defined as: $card\{\mathbf{M} \in \mathrm{NExt}(\mathbf{L}_0) \mid \mathbf{L} \equiv_K \mathbf{M}\}$. $\mathbf{L}$ is called *intrinsically complete* over $\mathbf{L}_0$ if $\delta_{\mathbf{L}_0}(\mathbf{L}) = 1$.

On the maps appeared in this paper, σ's (σ, σ_k's, and σ_∞) preserve the Kripke completeness and the finite model property. For example, σ_k maps $\mathbf{L}$ to $\mathbf{L} \cap \mathbf{L}(\mathcal{I}_k)$. Since the frame $\mathcal{I}_k$ is a finite Kripke frame, if $\mathcal{L}$ is Kripke complete (and also has the f.m.p), then so is $\mathbf{L} \cap \mathbf{L}(\mathcal{I}_k)$. (and also with f.m.p). Although $\mathcal{I}_\infty$ is not a finite Kripke frame, it can be said that the logic $\mathbf{L}(\mathcal{I}_\infty)$ has the finite model property. Thus the same argument goes through in the case of σ_∞. Then, to the converse, is it the case that τ's (τ, τ_k's and τ_∞) preserve the intrinsically completeness?

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