Kripke Completeness of Strictly Positive Modal Logics Over Meet Semilattices with Operators

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This talk is about a connection between various consequence relations for the fragment of propositional multi-modal logic that comprises implications $\sigma \to \tau$, where σ and τ are *strictly positive modal formulas* constructed from propositional variables using conjunction $\wedge$, unary diamond operators $\Diamond_i$, and the constant 'truth' $\top$. We call such formulas SP-*implications* and we call sets of SP-implications SP-*theories*. These formulas can be interpreted on Kripke frames as well as on meet semilattices with unary monotone operators (SLOs). This gives rise to the following problem:

(***completeness***) identify SP-theories P that are *complete* in the sense that the two consequence relations $P \models_{\mathsf{Kr}}$ and $P \models_{\mathsf{SLO}}$ coincide, where for any SP-implication ι,

$$P \models_{\mathsf{Kr}} \iota \quad \text{iff} \quad \iota \text{ is valid in every Kripke frame validating } P;$$
$$P \models_{\mathsf{SLO}} \iota \quad \text{iff} \quad \iota \text{ is valid in every SLO validating } P.$$

SP-implications are Sahlqvist, so for every modal formula φ and SP-theory P

$$P \models_{\mathsf{Kr}} \varphi \quad \text{iff} \quad \varphi \in \mathbf{K} + P \quad \text{iff} \quad \varphi \approx \top \text{ is valid in every BAO validating } P, \qquad (1)$$

where BAO stands for *Boolean algebras with normal and* $\vee$*-additive unary operators*. Note that, by (1), the completeness problem is equivalent to

(***conservativity***) the purely algebraic problem of whether the consequence relation $P \models_{\mathsf{BAO}}$ is *conservative* over $P \models_{\mathsf{SLO}}$ with respect to SP-implications, that is, $P \models_{\mathsf{SLO}} \iota$ iff $P \models_{\mathsf{BAO}} \iota$, for any ι; and also to

(***axiomatisability***) the problem whether P *axiomatises* the SP-implicational fragment of the normal modal logic $\mathbf{K} + P$ using the syntactic Birkhoff-type calculus corresponding to the algebraic consequence relation $P \models_{\mathsf{SLO}}$ (in other words, the problem whether P has a *modal companion*).

I am going to present two methods for proving completeness for SP-theories together with numerous sufficient conditions for their applicability. Note that incomplete SP-theories are easy to find, with two simplest ones being $\{\Diamond p \to p\}$ and $\{\Diamond p \to \Diamond q\}$.

This talk is based on a recent joint work with Agi Kurucz, Yoshihito Tanaka, Frank Wolter and Michael Zakharyaschev accessible at `https://arxiv.org/pdf/1708.03403.pdf`