PSPACE-decidability of Japaridze's polymodal logic

Ilya Shapirovsky

ABSTRACT. In this paper we prove that Japaridze's Polymodal Logic is PSPACE-decidable. To show this, we describe a decision procedure for satisfiability on *hereditarily ordered frames* that can be applied to obtain upper complexity bounds for various modal logics.

Keywords: Japaridze's polymodal Logic, computational complexity, conditional satisfiability

1 Introduction

In this paper we investigate the complexity of well-known *propositional polymodal provability logic* GLP. This logic was introduced in [8], and now plays a significant role in proof theory (see [4, 1]). In [8, 7], the logic GLP was proved to be decidable, the question about its complexity was left open. The PSPACE-decidability of GLP was conjectured in the recent paper [2].

The logic GLP is known to be Kripke-incomplete. In [2], it was shown that GLP is polynomial-time reducible to a logic J with an explicit Kripke semantics: J is characterized by a class of finite *hereditary orders*. This class is defined as follows: a strict partial order is a hereditary order; a strictly ordered (by a new relation, which is also a strict partial order) set of hereditary orders is a hereditary order.

This paper proves the PSPACE-decidability of J. We propose a technique allowing us to check modal satisfiability on frames obtained by "hereditarily ordering". This approach seems to be applicable to a large class of transitive logics: in section 4 we give semantical conditions, sufficient for PSPACE-decidability (Theorems 21 and 22 for the monomodal case, Theorem 35 for the multi-modal case).

The paper is organized as follows. Section 2 introduces some standard notions and notations. In section 3 we describe some truth-preserving transformations for ordered sets of frames. In section 4, we introduce a notion of *conditional satisfiability*, and show, how it can be applied to obtain decision procedures for satisfiability on hereditarily ordered frames. First we formulate it for the monomodal case and then generalize for the multi-modal case. In section 5, we apply the described technique to obtain a PSPACE-decision procedure for J, and thus for GLP.

2 Preliminaries

We consider *propositional normal modal logics* with finitely or countably many modalities. Modal formulas are built using the connectives $\bot$ (*false*), $\to$ (*implication*), a countable set of unary connectives $\Diamond_1, \Diamond_2, \dots$ (*diamonds*) and a countable set of *propositional variables* $PV = \{p_1, p_2, \dots\}$. All other connectives are defined in the standard way, in particular $\Box_i\psi = \neg\Diamond_i\neg\psi$. An N-formula is a formula that contains only connectives $\Diamond_1, \dots, \Diamond_N, \to, \bot$.

An N-frame F is a tuple $(W, R_1, \dots, R_N)$, where $W \neq \varnothing$, $R_1, \dots, R_N \subseteq W \times W$; $R_1, \dots, R_N$ are called *accessability relations of* F.

In this paper we assume that all considered accessability relations are *transitive*.

An N-model M over a frame F is a pair (F, θ), where $\theta : PV \to 2^W$. The notations $w \in \mathsf{M}$, $w \in \mathsf{F}$ mean $w \in W$.

A *weak submodel* M' of M is a model $((W', R'_1, \dots, R'_N), \eta)$, such that $W' \subseteq W$, $R'_1 \subseteq R_1, \dots, R'_N \subseteq R_N$, and $\eta(p) = \theta(p) \cap W'$ for any $p \in PV$.

For $R \subseteq W \times W$, $V \subseteq W$, by $R|V$ we denote the restriction R to V: $R|V = R \cap (V \times V)$. For an N-frame $\mathsf{F} = (W, R_1, \dots, R_N)$, by $\mathsf{F}|V$ we denote the *restriction* F *to* V: $\mathsf{F}|V = (V, R_1|V, \dots, R_N|V)$. If M is a model over F, and G is the restriction F to V, then the submodel of M over G is called the *restriction* M *to* V *(to* G*)*, in symbols, $\mathsf{M}|\mathsf{G}$ or $\mathsf{M}|V$.

The *trues of a formula at a point in a model*, and also the *validity of a formula in a frame* (*in a class of frames*) are defined in the standard way, see e.g. [3]; in symbols, $\mathsf{M}, w \vDash \varphi$ means that φ is true at w in M, $\mathsf{F} \vDash \varphi$ means that φ is valid in F. Also, for a set of formulas Ψ, $\mathsf{F} \vDash \Psi$ means $\mathsf{F} \vDash \varphi$ for any $\varphi \in \Psi$.

For an N-frame F, an N-formula φ is *satisfiable in* F (or F-*satisfiable*), if φ is true at some point of a model over F. For a class of frames $\mathcal{F}$, φ is *satisfiable in* $\mathcal{F}$ (or $\mathcal{F}$-*satisfiable*), if φ is F-satisfiable for some $\mathsf{F} \in \mathcal{F}$. For a logic L, φ is L-*satisfiable*, if φ is F-satisfiable for some $\mathsf{F} \vDash \mathrm{L}$.

As usual, a *cluster* in a frame (W, R) is an $\sim_R$-equivalence class, where $\sim_R \;=\; (R \cap R^{-1}) \cup \{(w, w) \mid w \in W\}$. Also, by a *cluster* we mean a frame $\mathsf{F} = (W, W \times W)$, or a frame $(\{w\}, \varnothing)$ (*degenerate cluster*). For $n \geq 1$, C_n denotes the n-element cluster $(W_n, W_n \times W_n)$, where $W_n = \{1, \dots, n\}$; $\mathsf{C}_0 = (\{0\}, \varnothing)$.

For a frame $\mathsf{F} = (W, R)$, $w \in W$, put $R(w) = \{w' \mid wRw'\}$. If $W = R(w) \cup \{w\}$, then F is a *cone* (or *rooted frame*), and w is called a *root* of F.

For N-frames F and G, the notation $g : \mathsf{F} \twoheadrightarrow \mathsf{G}$ means that g is a p-morphism from F onto G; $\mathsf{F} \twoheadrightarrow \mathsf{G}$ means that $g : \mathsf{F} \twoheadrightarrow \mathsf{G}$ for some g. Recall that if $\mathsf{F} \twoheadrightarrow \mathsf{G}$ then any G-satisfiable formula is F-satisfiable (see e.g. [3]).

Let us recall the notion of *selective filtration*.

DEFINITION 1. Let M be an N-model, Ψ a set of N-formulas closed under subformulas. A weak submodel M' of M is called a *selective filtration of* M *through* Ψ, if for any $w \in \mathsf{M}'$, for any formula ψ, for all $i = 1, \dots, N$, we

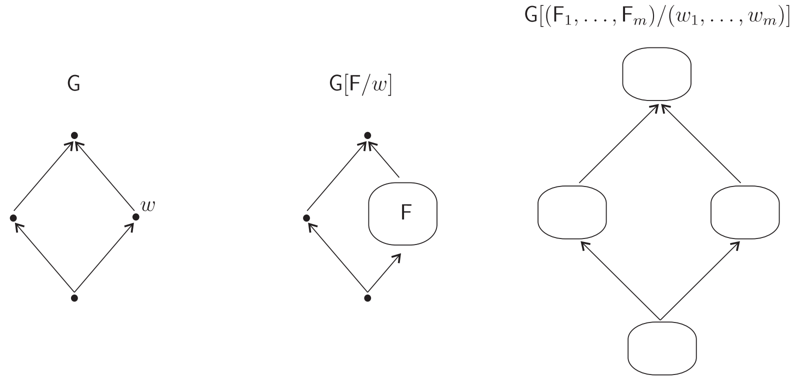

Figure 1.

have

$$\Diamond_i\psi \in \Psi \ \&\ \mathsf{M}, w \vDash \Diamond_i\psi \ \Rightarrow\ \exists u \in R'_i(x)\ \mathsf{M}, u \vDash \psi,$$

where $R'_1, \ldots, R'_N$ are the the accessability relations of M'.

LEMMA 2. *If* M' *is a selective filtration of* M *through* Ψ*, then for any* $w \in \mathsf{M}'$*, for any* $\psi \in \Psi$*, we have*

$$\mathsf{M}, w \vDash \psi \Leftrightarrow \mathsf{M}', w \vDash \psi.$$

For a modal formula φ, $Sub(\varphi)$ denotes the set of all subformulas of φ, $\langle\varphi\rangle$ denotes the cardinality of $Sub(\varphi)$.

On the set of all modal formulas we fix a linear order $\lessdot$, such that for any ϕ, ψ, if $\phi \in Sub(\psi)$ then $\phi \lessdot \psi$. For a set of formulas Ψ, let $\Psi_\lessdot$ denote the list of elements of Ψ ordered by $\lessdot$. If $\Psi_\lessdot = (\psi_1, \ldots, \psi_n)$, then for a boolean vector $\mathbf{v} = (v_1, \ldots, v_n) \in \{0,1\}^n$ we put

$$\Psi_{\mathbf{v}} = \{\psi_i \mid v_i = 1,\ 1 \le i \le n\},\quad \Psi^{\mathbf{v}} = \bigwedge_{1\le i\le n} \psi_i^{v_i},$$

where $\psi^1 = \psi$ and $\psi^0 = \neg\psi$.

3 Partially ordered sets of frames

It is well-known that any transitive frame can be viewed as a set of clusters ordered by a transitive and antisymmetric relation (*skeleton*, see e.g. [6]). The following construction allows us to consider arbitrary frames instead of clusters.

DEFINITION 3. Let $\mathsf{G} = (W, R)$ be a finite (strict or non-strict) partial order, $m = |W|$, $W = \{w_1, \ldots, w_m\}$.

For frames $\mathsf{F}_1 = (V_1, S_1), \ldots, \mathsf{F}_m = (V_m, S_m)$, we define the frame $\mathsf{G}[(\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)] = (\overline{W}, \overline{R})$ obtained by *replacing points*

$w_1, \dots, w_m$ *with frames* $\mathsf{F}_1, \dots, \mathsf{F}_m$ (Fig. 1):

$$\overline{W} = (\{w_1\} \times V_1) \ \cup \ \dots \cup \ (\{w_m\} \times V_m),$$

$$(w', v')\overline{R}(w'', v'') \Leftrightarrow (w' \neq w'' \ \& \ w'Rw'') \ \text{ or } \ (w' = w'' = w_i \ \& \ v'S_iv'').$$

Also, for $w \in W$ and a frame $\mathsf{F} = (V, S)$, we define the frame $\mathsf{G}[\mathsf{F}/w] = (W', R')$ obtained by *replacing* w *with* F (Fig. 1):

$$W' = (W - \{w\}) \ \cup \ V', \ \text{where } V' = \{w\} \times V,$$

$$R' = R|(W - \{w\}) \ \cup \ \{((w, u'), (w, u'')) \mid u'Su''\} \cup$$
$$\cup (V' \times (R(w) - \{w\})) \ \cup \ ((R^{-1}(w) - \{w\}) \times V').$$

For a class $\mathcal{F}$ of monomodal frames, we put

$$\mathsf{G}[\mathcal{F}] = \{\mathsf{G}[(\mathsf{F}_1, \dots, \mathsf{F}_m)/(w_1, \dots, w_m)] \mid \mathsf{F}_1, \dots, \mathsf{F}_m \in \mathcal{F}\}.$$

Finally, for a class $\mathcal{G}$ of finite partial orders, we put

$$\mathcal{G}[\mathcal{F}] = \bigcup \{\mathsf{G}[\mathcal{F}] \mid \mathsf{G} \in \mathcal{G}\},$$

i.e. $\mathcal{G}[\mathcal{F}]$ is the class of frames, obtained from frames that belong to $\mathcal{G}$ by replacing all their points with frames from $\mathcal{F}$.

REMARK 4. By a straightforward argument,

$$\mathsf{G}[(\mathsf{F}_1, \dots, \mathsf{F}_m)/(w_1, \dots, w_m)] = \mathsf{G}[\mathsf{F}_1/w_1] \dots [\mathsf{F}_m/w_m].$$

Note that the frame $\mathsf{G}[\mathsf{F}/w]$ is transitive due to the transitivity of the relations R and S. Thus all frames that described in the above definition are transitive.

Note also that if G is strict, and G' is the corresponding non-strict partial order, then

$$\mathsf{G}[(\mathsf{F}_1, \dots, \mathsf{F}_m)/(w_1, \dots, w_m)] = \mathsf{G}'[(\mathsf{F}_1, \dots, \mathsf{F}_m)/(w_1, \dots, w_m)].$$

EXAMPLE 5. Let $\mathcal{PO}$ denote the class of all finite non-strict partial orders.

If $\mathcal{F} = \{\mathsf{C}_0\}$, then $\mathcal{PO}[\mathcal{F}]$ is the class of all finite strict partial orders, up to isomorphisms.

If $\mathcal{F}$ is the class of all finite (non-degenerate) clusters, then $\mathcal{PO}[\mathcal{F}]$ is the class of all finite transitive (and reflexive) frames, up to isomorphisms.

Let us generalize the above construction for the multi-modal case.

DEFINITION 6. Let $\mathsf{G} = (W, R) \in \mathcal{PO}$, $m = |W|$, $W = \{w_1, \dots, w_m\}$.

For an N-frame $\mathsf{F} = (V, S_1, \dots, S_N)$, $1 \leq k \leq N$, we define the frame $\mathsf{G}[k; \mathsf{F}/w] = (W', R'_1, \dots R'_N)$ as follows. To define W' and R'_k, we put $(W', R'_k) = \mathsf{G}[(V, S_k)/w]$; for $l \neq k$ we put

$$R'_l = \{((w, u'), (w, u'')) \mid u', u'' \in V, \ u'S_lu''\}.$$

For a class of N-frames $\mathcal{F}$, $1 \le k \le N$, we put

$$\mathsf{G}[k;\mathcal{F}] = \{\mathsf{G}[k;\mathsf{F}_1/w_1]\dots[k;\mathsf{F}_m/w_m] \mid \mathsf{F}_1,\dots,\mathsf{F}_m \in \mathcal{F}\},$$

and for a class $\mathcal{G}$ of finite partial orders,

$$\mathcal{G}[k;\mathcal{F}] = \bigcup\{\mathsf{G}[k;\mathcal{F}] \mid \mathsf{G} \in \mathcal{G}\}.$$

PROPOSITION 7. *Let* $\mathsf{G} \in \mathcal{PO}$, $w \in \mathsf{G}$, F *and* F' *be* N*-frames,* $1 \le k \le N$. *Then* $\mathsf{F}' \twoheadrightarrow \mathsf{F}$ *implies* $\mathsf{G}[k;\mathsf{F}'/w] \twoheadrightarrow \mathsf{G}[k;\mathsf{F}/w]$.

Proof. Let $g : \mathsf{F}' \twoheadrightarrow \mathsf{F}$. The required p-morphism g' is defined as follows. For $v \in \mathsf{F}'$ put $g'(w,v) = (w, g(v))$; for $w' \in \mathsf{G} - \{w'\}$ put $g'(w') = w'$. ■

Let $\mathsf{F} \in \mathcal{PO}$. By $Ht(\mathsf{F})$ we denote the height of F, i.e., the maximal length of strictly ascending chains in F (by length of a chain we mean the number of its elements); by branching of a point $w \in \mathsf{F}$ we mean the number of immediate successors of w in F; $Br(\mathsf{F})$ denotes the branching of F, i.e., the maximal branching of its points.

By a *tree* we mean a rooted non-strict partial order (W,R) such that $R^{-1}(w)$ is a chain for every w. By $\mathcal{T}$ we denote the class of all finite trees. By $\mathcal{T}_{n,b}$ we denote the class of trees with the height not more then h and the branching not more then b:

$$\mathcal{T}_{h,b} = \{\mathsf{T} \in \mathcal{T} \mid Ht(\mathsf{T}) \le h,\ Br(\mathsf{T}) \le b\}.$$

Let us recall the notions of *disjoint sum* (or *disjoint union*) and *ordinal sum* of frames. Suppose that frames $\mathsf{F}_1 = (W_1,R_1)$ and $\mathsf{F}_2 = (W_2,R_2)$ have no common points. Put

$\mathsf{F}_1 \sqcup \mathsf{F}_2 = (W_1 \cup W_2, R_1 \cup R_2)$	*disjoint sum of* F_1 *and* F_2,
$\mathsf{F}_1 + \mathsf{F}_2 = (W_1 \cup W_2, R_1 \cup (W_1 \times W_2) \cup R_2)$	*ordinal sum of* F_1 *and* F_2.

If M is a model over the frame $\mathsf{F}_1 \sqcup \mathsf{F}_2$ (over the frame $\mathsf{F}_1 + \mathsf{F}_2$), and $\mathsf{M}_1 = \mathsf{M}|W_1$, $\mathsf{M}_2 = \mathsf{M}|W_2$, then M is called the *disjoint (ordinal) sum of models* M_1 and M_2, in symbols: $\mathsf{M} = \mathsf{M}_1 \sqcup \mathsf{M}_2$ ($\mathsf{M} = \mathsf{M}_1 + \mathsf{M}_2$).

REMARK 8. By sum of frames that have common points, we mean sum of their isomorphic copies:

$$\mathsf{F}_1 \sqcup \mathsf{F}_2 = \mathsf{G}_\sqcup[(\mathsf{F}_1,\mathsf{F}_2)/(1,2)],\ \text{where } \mathsf{G}_\sqcup = (\{1,2\},\varnothing);$$

$$\mathsf{F}_1 + \mathsf{F}_2 = \mathsf{G}_+[(\mathsf{F}_1,\mathsf{F}_2)/(1,2)],\ \text{where } \mathsf{G}_+ = (\{1,2\},\{(1,2)\}).$$

Disjoint sum of N-frames is defined analogously (see e.g. [3]). Let us modify the notion of ordinal sum for the multi-modal case.

DEFINITION 9. Let $\mathsf{F} = (W,R_1,\dots,R_N)$ and $\mathsf{F}' = (W',R'_1,\dots,R'_N)$ be N-frames, $1 \le k \le N$. Put $\mathsf{F} +_k \mathsf{F}' = (V,S_1,\dots,S_N)$, where $(V,S_k) = (W,R_k) + (W',R'_k)$, and $(V,S_l) = (W,R_l) \sqcup (W',R'_l)$ for $l \ne k$.

The above definitions imply the following

PROPOSITION 10. *Let $\mathcal{F}$ be a class of N-frames, $1 \leq k \leq N$, and let $\mathsf{G} \in \mathcal{T}_{h+1,b}[k;\mathcal{F}]$ for some $h, b \geq 1$. Then G is either isomorphic to a frame $\mathsf{F} \in \mathcal{F}$ or isomorphic to a frame $\mathsf{F} +_k (\mathsf{G}_1 \sqcup \cdots \sqcup \mathsf{G}_{b'})$, where $1 \leq b' \leq b$, $\mathsf{F} \in \mathcal{F}$, $\mathsf{G}_1, \dots \mathsf{G}_{b'} \in \mathcal{T}_{h,b}[k;\mathcal{F}]$.*

Proof. For some $\mathsf{T} \in \mathcal{T}_{h+1,b}$, we have $\mathsf{G} \in \mathsf{T}[k;\mathcal{F}]$. Then either T is a singleton (when $Ht(\mathsf{T}) = 1$), and in this case G is isomorphic to a frame $\mathsf{F} \in \mathcal{F}$, or T is isomorphic to a frame $\mathsf{C}_1 + (\mathsf{T}_1 \sqcup \cdots \sqcup \mathsf{T}_{b'})$, where b' is the branching at the root of T and $\mathsf{T}_1, \dots \mathsf{T}_{b'} \in \mathcal{T}_{h,b}$. ■

LEMMA 11. *Let $\mathcal{F}$ be a class of N-frames, $1 \leq k \leq N$, G be a finite rooted partial order. Then for any $\mathsf{H} \in \mathsf{G}[k;\mathcal{F}]$ there exists a tree $\mathsf{T} \in \mathcal{T}$ such that for some $\mathsf{H}' \in \mathcal{T}[k;\mathcal{F}]$ we have $\mathsf{H}' \twoheadrightarrow \mathsf{H}$.*

Proof. By the standard unravelling argument. Let w_0 be the root of G. To define $\mathsf{T} = (W, R)$, put

$W = \{(w_0, \dots, w_k) \mid w_0, \dots, w_k \in W,\ \ w_{i+1}$ is an immediate successor of w_i for all $i = 0, \dots, k-1\}$;

$(w_0, \dots, w_k) R (w_0, \dots, w_l) \Leftrightarrow (w_0, \dots, w_k)$ is a prefix of $(w_0, \dots, w_l)$. ■

It is well-known that any K4-satisfiable formula φ is satisfiable in some finite frame with the height and the branching of its skeleton not more then $\langle\varphi\rangle$ (see e.g. [6]). The following lemma generalizes this observation.

LEMMA 12. *Let $\mathcal{F}$ be a class of N-frames, $1 \leq k \leq N$. If an N-formula φ is $\mathcal{PO}[k;\mathcal{F}]$-satisfiable, then φ is $\mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}[k;\mathcal{F}]$-satisfiable.*

Proof. By Lemma 11, φ is satisfiable in a frame $\mathsf{H} \in \mathsf{T}[k;\mathcal{F}]$, where $\mathsf{T} \in \mathcal{T}$. Then $\mathsf{M}, (w_0, v) \vDash \varphi$, where M is a model over H, $w_0 \in \mathsf{T}$, $(w_0, v) \in \mathsf{H}$.

Let $\mathsf{T} = (W, R)$. For a point $w \in W$, put

$$\Psi_w = \{\Diamond_k \psi \in Sub(\varphi) \mid \mathsf{M}, (w, v) \vDash \Diamond_k \psi \text{ for some } (w, v) \in \mathsf{H}\}.$$

Inductively we define a set W_i. Put $W_0 = \{w_0\}$, $\Psi_i = \bigcup\{\Psi_w \mid w \in W_i\}$.

If $\Psi_i \neq \varnothing$, we define W_{i+1}. First, for every $w \in W_i$ we define a set U_w: if $\Psi_w = \varnothing$, we put $U_w = \varnothing$; for $\Psi_w = \{\Diamond_k \psi_1, \dots, \Diamond_k \psi_l\}$, put $U_w = \{u_1, \dots, u_l\}$, where u_i is an R-maximal point in the set

$$\{u \mid u \in R(w),\ \mathsf{M}, (u, v) \vDash \psi_i \text{ for some } (u, v) \in \mathsf{M}\}.$$

Put $W_{i+1} = \bigcup\{U_w \mid w \in W_i\}$.

Note that $|\Psi_{i+1}| < |\Psi_i|$, thus for some $l < \langle\varphi\rangle$ we obtain $\Psi_l = \varnothing$. Then we put $W' = W_0 \cup \cdots \cup W_l$, $V' = \{(w, v) \in \mathsf{H} \mid w \in W'\}$, and put $\mathsf{T}' = \mathsf{T}|W'$, $\mathsf{M}' = \mathsf{M}|V'$, $\mathsf{H}' = \mathsf{H}|V'$. Due to the construction, $\mathsf{T}' \in \mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}$ and $\mathsf{H}' \in \mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}[k;\mathcal{F}]$. Also, M' is a selective filtration of M through $Sub(\varphi)$, thus φ is $\mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}[k;\mathcal{F}]$-satisfiable. ■

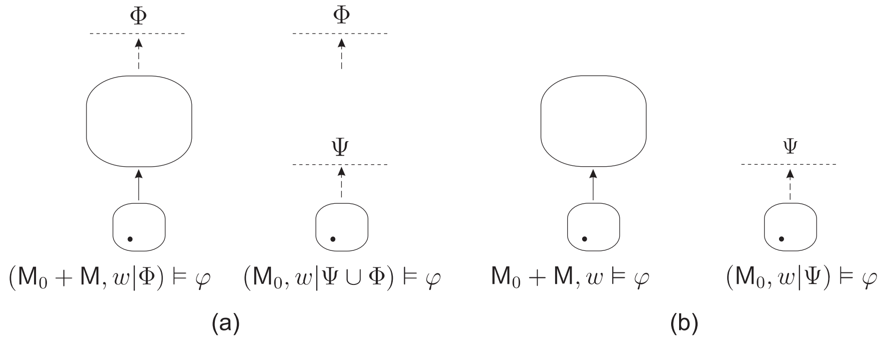

Figure 2.

4 Conditional satisfiability

4.1 Monomodal case

Is this subsection we assume that all frames, models and formulas are monomodal.

DEFINITION 13. Let M be a model, Ψ be a set of formulas. For a formula φ and a point $w \in \mathsf{M}$, we define the truth-relation $(\mathsf{M}, w|\Psi) \vDash \varphi$:

$$\begin{array}{lcl}
(\mathsf{M}, w|\Psi) \vDash p & \Leftrightarrow & \mathsf{M}, w \vDash p \\
(\mathsf{M}, w|\Psi) \nvDash \bot & & \\
(\mathsf{M}, w|\Psi) \vDash \varphi \to \psi & \Leftrightarrow & (\mathsf{M}, w|\Psi) \nvDash \varphi \text{ or } (\mathsf{M}, w|\Psi) \vDash \psi \\
(\mathsf{M}, w|\Psi) \vDash \Diamond\varphi & \Leftrightarrow & \varphi \in \Psi \text{ or } \Diamond\varphi \in \Psi \text{ or} \\
& & \text{for some } v \in R(w) \text{ we have } (\mathsf{M}, v|\Psi) \vDash \varphi,
\end{array}$$

where R is the accessability relation in M.

We read $(\mathsf{M}, w|\Psi) \vDash \varphi$ as "*φ is true at w in M under the condition Ψ*".

Note that $(\mathsf{M}, w|\varnothing) \vDash \varphi \Leftrightarrow \mathsf{M}, w \vDash \varphi$.

PROPOSITION 14. *Consider models M_0, M, their ordinal sum $\mathsf{M}_0 + \mathsf{M}$, and a set of formulas Φ (Fig. 2,a). If*

$$\Psi = \{\psi \in Sub(\varphi) \mid (\mathsf{M}, v|\Phi) \vDash \psi \textit{ for some } v\},$$

then for any formula φ, $w \in \mathsf{M}_0$,

$$(\mathsf{M}_0 + \mathsf{M}, w|\Phi) \vDash \varphi \Leftrightarrow (\mathsf{M}_0, w|\Psi \cup \Phi) \vDash \varphi.$$

Proof. The proof is straightforward, by induction on the length of φ. Consider only the case $\varphi = \Diamond\psi$, $\psi \notin \Phi$, $\Diamond\psi \notin \Phi$.

Suppose $(\mathsf{M}_0+\mathsf{M}, w|\Phi) \vDash \Diamond\psi$. Then $(\mathsf{M}_0+\mathsf{M}, v|\Phi) \vDash \psi$ for some $v \in R(w)$, where R is the accessability relation of $\mathsf{M}_0 + \mathsf{M}$. If $v \in \mathsf{M}$, then $\psi \in \Psi$; if $v \in \mathsf{M}_0$, then $(\mathsf{M}_0, v|\Psi \cup \Phi) \vDash \psi$ by the induction hypothesis; in both cases $(\mathsf{M}_0, w|\Psi \cup \Phi) \vDash \Diamond\psi$.

Suppose $(\mathsf{M}_0, w|\Psi\cup\Phi) \vDash \Diamond\psi$. If $\psi \in \Psi$ or $\Diamond\psi \in \Psi$, then $(\mathsf{M}, v|\Phi) \vDash \psi\vee\Diamond\psi$ for some $v \in \mathsf{M}$. Thus $(\mathsf{M}_0 + \mathsf{M}, w|\Phi) \vDash \Diamond\psi$. ■

COROLLARY 15. *Consider models* M_0 *and* M. *For any formula* φ, $w \in \mathsf{M}_0$, *we have (Fig. 2,b)*

$$\mathsf{M}_0 + \mathsf{M}, w \vDash \varphi \Leftrightarrow (\mathsf{M}_0, w|\{\psi \in Sub(\varphi) \mid \mathsf{M}, v \vDash \psi \text{ for some } v\}) \vDash \varphi.$$

DEFINITION 16. Let F be a cone, Ψ be a set of formulas. We say that φ is F-*satisfiable under the condition* Ψ, if φ is true at a root of F in some model over F under the condition Ψ. For a formula φ and vectors $\mathbf{v}, \mathbf{u} \in \{0,1\}^{\langle\varphi\rangle}$, the notation

$$\mathsf{F} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$$

means that the formula $Sub(\varphi)^{\mathbf{v}}$ is F-satisfiable under the condition $Sub(\varphi)_{\mathbf{u}}$. For a class $\mathcal{F}$ of cones, $\mathcal{F} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$ means that $\mathsf{F} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$ for some $\mathsf{F} \in \mathcal{F}$.

The following constructions are generalization of the construction proposed in [10].

DEFINITION 17. For a positive integer d, a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of sets of cones is called *d-moderate*, if there exists an algorithm such that for any formula φ and any vectors $\mathbf{u}, \mathbf{v} \in \{0,1\}^{\langle\varphi\rangle}$ it decides whether

$$\mathcal{F}_{\langle\varphi\rangle} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$$

in space $O(\langle\varphi\rangle^d)$. A sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is *moderate*, if it is d-moderate for some integer d.

EXAMPLE 18. It is clear, that if a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ can be effectively described in polynomial of n space, then it is moderate. In particular, if $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a sequence of finite sets of finite cones, such that for some k $\mathcal{F}_k = \mathcal{F}_{k+1} = \dots$, then $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is moderate. For instance, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is moderate, if:

- $\mathcal{F}_n$ is the set of all (non-degenerate) clusters with cardinality not more then n: for all n $\mathcal{F}_n = \{\mathsf{C}_0, \dots, \mathsf{C}_n\}$ or for all n $\mathcal{F}_n = \{\mathsf{C}_1, \dots, \mathsf{C}_n\}$;
- $\mathcal{F}_n$ consists of a single frame which is a singleton: for all n $\mathcal{F}_n = \{\mathsf{C}_0\}$ or for all n $\mathcal{F}_n = \{\mathsf{C}_1\}$.

Next we show that tree-like structures "constructed" from moderate sequences are also moderate.

For boolean vectors $\mathbf{u}, \mathbf{v}$ of the same length, let $\mathbf{u} \vee \mathbf{v}$ denote their bitwise disjunction.

PROPOSITION 19. *Let* $\mathcal{F}$ *be a class of cones. Then for any formula* φ, *for any* $\mathbf{u}, \mathbf{v} \in \{0,1\}^{\langle\varphi\rangle}$, *for any integers* $h, b \geq 1$, *the following two conditions are equivalent.*

1. *$\mathcal{T}_{h+1,b}[\mathcal{F}] \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$.*

2. *Either $\mathcal{F} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$, or for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{b'} \in \{0,1\}^{\langle\varphi\rangle}$, where $1 \leq b' \leq b$, we have: $\mathcal{T}_{h,b}[\mathcal{F}] \mid \mathbf{u} \Vdash_\varphi \mathbf{v}_j$ for all $j = 1, \ldots, b'$, and $\mathcal{F} \mid \mathbf{u} \vee \mathbf{v}_1 \cdots \vee \mathbf{v}_{b'} \Vdash_\varphi \mathbf{v}$.*

Proof. Put $\Phi = Sub(\varphi)$.

($1 \Rightarrow 2$) Suppose that $\mathcal{T}_{h+1,b}[\mathcal{F}] \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$. Then $\mathsf{G} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$ for some $\mathsf{G} \in \mathsf{T}[\mathcal{F}]$, where $\mathsf{T} \in \mathcal{T}$, $Ht(\mathsf{T}) = h' \leq h+1$ and $Br(\mathsf{T}) \leq b$.

The case $h' = 1$ is trivial: here G is isomorphic to some frame from $\mathcal{F}$, thus $\mathcal{F} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$.

Suppose $h' > 1$. Let b' be the branching at the root of T. Then G is isomorphic to a frame $\mathsf{F} + (\mathsf{G}_1 \sqcup \cdots \sqcup \mathsf{G}_{b'})$, where $\mathsf{F} \in \mathcal{F}$, $1 \leq b' \leq b$, $\mathsf{G}_1, \ldots \mathsf{G}_{b'} \in \mathcal{T}_{h,b}[\mathcal{F}]$.

For some model M over G we have $(\mathsf{M}, w|\Phi_{\mathbf{u}}) \vDash \Phi^{\mathbf{v}}$, where w is a root of G. For $1 \leq j \leq b'$, let w_j be a root of G_j,

$$\Phi_j = \{\psi \in \Phi \mid (\mathsf{M}, w_j|\Phi_{\mathbf{u}}) \vDash \psi\}.$$

Then $\mathsf{G}_j \mid \mathbf{u} \Vdash_\varphi \mathbf{v}_j$, where $\mathbf{v}_j$ is determined by the equation $\Phi_j = \Phi_{\mathbf{v}_j}$.

For a formula $\psi \in \Phi$, we have:

$$(\mathsf{M}, w'|\Psi_{\mathbf{u}}) \vDash \psi \text{ for some } w' \in \mathsf{G}_1 \sqcup \cdots \sqcup \mathsf{G}_{b'} \text{ iff } \psi \in \Phi_j \text{ for some } j.$$

Let M' be the restriction M to F. By Proposition 14, we obtain $(\mathsf{M}', w|\Psi_{\mathbf{u}} \cup \Phi_1 \cup \cdots \cup \Phi_{b'}) \vDash \Phi^{\mathbf{v}}$. Thus $\mathcal{F} \mid \mathbf{u} \vee \mathbf{v}_1 \cdots \vee \mathbf{v}_{b'} \Vdash_\varphi \mathbf{v}$.

($2 \Rightarrow 1$) If $\mathsf{F} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$ for some $\mathsf{F} \in \mathcal{F}$, then $\mathcal{T}_{h+1,b}[\mathcal{F}] \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$, since F is isomorphic to some frame from $\mathcal{T}_{1,1}[\mathcal{F}]$.

In the second case, $(\mathsf{M}', w|\Psi_{\mathbf{u}} \cup \Phi_{\mathbf{v}_1} \cup \cdots \cup \Phi_{\mathbf{v}_{b'}}) \vDash \Phi^{\mathbf{v}}$ for some model M' over a frame $\mathsf{F} \in \mathcal{F}$, and $(\mathsf{M}_j, w_j|\Phi_u) \vDash \Psi^{\mathbf{v}_j}$ for some models M_j over frames from $\mathcal{T}_{h,b}[\mathcal{F}]$, where w is a root of M, w_j is a root of M_j, $j = 1, \ldots, b'$. Put $\mathsf{M} = \mathsf{M}' + (\mathsf{M}_1 \sqcup \cdots \sqcup \mathsf{M}_{b'})$. By Proposition 14, we have $(\mathsf{M}, w|\Psi_{\mathbf{u}}) \vDash \Phi^{\mathbf{v}}$. Thus $\mathcal{T}_{h+1,b}[\mathcal{F}] \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$. ■

COROLLARY 20. *Let $\mathcal{F}$ be a class of cones. Suppose that* SatModerate *is an algorithm such that for any formula φ, for any $\mathbf{u}, \mathbf{v} \in \{0,1\}^{\langle\varphi\rangle}$, it decides whether*

$$\mathcal{F} \mid \mathbf{u} \Vdash_\varphi \mathbf{v}.$$

Then SatTree *(see Table 1) is an algorithm such that for any formula φ, for any $\mathbf{u}, \mathbf{v} \in \{0,1\}^{\langle\varphi\rangle}$, for any integers $h, b \geq 1$, it decides whether*

$$\mathcal{T}_{h,b}[\mathcal{F}] \mid \mathbf{u} \Vdash_\varphi \mathbf{v}.$$

THEOREM 21. *If $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is d-moderate sequence of sets of cones, and P is a polynomial of degree d', then the sequence $(\mathcal{T}_{P(n),P(n)}[\mathcal{F}_n])_{n\in\mathbb{N}}$ is $\max\{2+d', d\}$-moderate.*

Table 1. Algorithm SatTree

Function SatTree(*formula* φ; *boolean vectors* $\mathbf{v}, \mathbf{u}$; *integers* h, b)
returns boolean;
* SatTree *decides whether* $\mathcal{T}_{h,b}[\mathcal{F}] \mid \mathbf{u} \Vdash_\varphi \mathbf{v}$ *\

```
Begin
 if SatModerate(φ, v, u) then
   \*  F | u ⊩_φ v  *\
   return(true);
 if h > 1 then
   for every integer b' such that 1 ≤ b' ≤ b
     \*  b' is the branching  *\
     for every boolean vectors v_1, ..., v_b' ∈ {0,1}^⟨φ⟩
       if SatModerate(φ, v, u ∨ v_1 ··· ∨ v_b') then
         \*  F | u ∨ v_1 ··· ∨ v_b' ⊩_φ v  *\
         if ⋀_{1≤j≤b'} SatTree(φ, v_j, u, h − 1, b) then
         \*  T_{h,b}[F] | u ⊩_φ v_j for all j  *\
           return(true);
 return(false);
End.
```

Proof. At every step of recursion, the algorithm SatTree uses $O(n^2)$ amount of space for a formula φ, where $n = \langle\varphi\rangle$. We also need $O(n^d)$ amount of space that used by SatModerate. The depth of recursion is $P(n)$, thus we need $O(n^2 P(n) + n^d)$ amount of space. ■

The above fact implies the following

THEOREM 22. *Suppose that a logic* L *is characterized by* $\mathcal{PO}[\mathcal{F}]$ *for some class* $\mathcal{F}$. *If there exists a moderate sequence* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *such that* $\mathcal{F}_n \subseteq \mathcal{F}$ *for all* $n \in \mathbb{N}$, *and any* L*-satisfiable formula* φ *is* $\mathcal{PO}[\mathcal{F}_{\langle\varphi\rangle}]$*-satisfiable, then* L *is in* PSPACE.

Proof. Consider an L-satisfiable formula φ with $\langle\varphi\rangle = n$. Then φ is satisfiable at a root of some $\mathsf{G} \in \mathcal{PO}[\mathcal{F}_n]$. By Lemma 12, φ is satisfiable at a root of some $\mathsf{G}' \in \mathcal{T}_{n,n}[\mathcal{F}_n]$. Thus, by Corollary 20, φ is L-satisfiable iff for some $\mathbf{v} = \{v_1, \ldots, v_{n-1}, 1\} \in \{0,1\}^n$ we have $\mathrm{SatTree}(\varphi, \mathbf{v}, (0, \ldots, 0), n, n) = true$. ■

COROLLARY 23. *If* $\mathrm{L} = \mathrm{L}(\mathcal{PO}[\mathcal{F}])$ *for some finite class of finite cones* $\mathcal{F}$, *then* L *is in PSPACE.*

Proof. Put $\mathcal{F}_n = \mathcal{F}$ for all n. ■

EXAMPLE 24. As an example, consider the logics K4, S4 , Gödel-Löb logic GL, and Grzegorczyk logic GRZ. They are well-known to be PSPACE-decidable, see e.g [9, 11, 5]. Let us illustrate, how this fact follows from Theorem 22.

GRZ (GL) is the logic of all finite non-strict (strict) partial orders, see e.g. [6]: $\mathrm{GRZ} = \mathrm{L}(\mathcal{PO}[\{C_1\}])$, $\mathrm{GL} = \mathrm{L}(\mathcal{PO}[\{C_0\}])$. By corollary 23, GRZ and GL are in PSPACE.

Note that any K4-satisfiable formula is satisfiable at some finite transitive frame F such that the cardinality of any cluster in F does not exceed $\langle\varphi\rangle$. Put

$$\mathcal{F}_n^{\mathrm{K4}} = \{\mathsf{C}_0, \ldots, \mathsf{C}_n\}, \quad \mathcal{F}_n^{\mathrm{S4}} = \{\mathsf{C}_1, \ldots, \mathsf{C}_n\}.$$

Then for any φ we have:

$$\varphi \text{ is K4-satisfiable iff } \varphi \text{ is } \mathcal{PO}[\mathcal{F}^{\mathrm{K4}}_{\langle\varphi\rangle}]\text{-satisfiable,}$$

$$\varphi \text{ is S4-satisfiable iff } \varphi \text{ is } \mathcal{PO}[\mathcal{F}^{\mathrm{S4}}_{\langle\varphi\rangle}]\text{-satisfiable.}$$

Since the sequences $(\mathcal{F}_n^{\mathrm{K4}})_{n\in\mathbb{N}}$ and $(\mathcal{F}_n^{\mathrm{S4}})_{n\in\mathbb{N}}$ are moderate, then by Theorem 22, K4 and S4 are in PSPACE.

REMARK 25. Using the standard method of *translating of QBF-formula into modal logics* [9], it is not difficult to obtain PSPACE-hardness for logics of classes $\mathcal{PO}[\mathcal{F}]$, thus described in Theorem 22 logics are PSPACE-complete (for non-empty $\mathcal{F}$).

4.2 Multi-modal case

DEFINITION 26. Let M be an N-model. A *condition for* M is a tuple $\overline{\Psi} = (\Psi_1, \ldots, \Psi_N)$ of sets of N-formulas. For an N-formula φ and a point $w \in \mathsf{M}$, we define the truth-relation $(\mathsf{M}, w|\overline{\Psi}) \vDash \varphi$ ("*φ is true at w in* M *under the condition* $\overline{\Psi}$"):

$$\begin{array}{lll}
(\mathsf{M}, w|\overline{\Psi}) \vDash p & \Leftrightarrow & \mathsf{M}, w \vDash p \\
(\mathsf{M}, w|\overline{\Psi}) \nvDash \bot & & \\
(\mathsf{M}, w|\overline{\Psi}) \vDash \varphi \to \psi & \Leftrightarrow & (\mathsf{M}, w|\Psi) \nvDash \varphi \text{ or } (\mathsf{M}, w|\Psi) \vDash \psi \\
(\mathsf{M}, w|\overline{\Psi}) \vDash \Diamond_k \varphi & \Leftrightarrow & \varphi \in \Psi_k \text{ or } \Diamond_k\varphi \in \Psi_k \text{ or} \\
 & & \text{for some } v \in R_k(w) \text{ we have } (\mathsf{M}, v|\overline{\Psi}) \vDash \varphi,
\end{array}$$

where $R_1, \ldots, R_N$ are the accessability relations in M.

PROPOSITION 27. *Consider N-frames F and G. Let $1 \le k \le N$, M be a model over the frame $\mathsf{F} +_k \mathsf{G}$, and let M' be the restriction M to F. Let φ be an N-formula φ, $\overline{\Phi}$ be a condition.*

Then for any $w \in \mathsf{M}'$ we have

$$(\mathsf{M}, w|\overline{\Phi}) \vDash \varphi \Leftrightarrow (\mathsf{M}', w|(\Phi'_1, \ldots, \Psi'_N)) \vDash \varphi,$$

where $\Phi'_i = \Phi_i$ for $i \neq k$, and

$$\Phi'_k = \Phi_k \cup \{\psi \in \mathit{Sub}(\varphi) \mid (\mathsf{M}, w|\overline{\Phi}) \vDash \psi \text{ for some } w \in \mathsf{G}\}.$$

Proof. By induction on the length of φ, analogously to the proof of Proposition 14. ■

DEFINITION 28. We call an N-frame $\mathsf{G} = (W, R_1, \ldots, R_N)$ *rooted*, if for some $w \in W$ we have $\{w\} \cup R_1(w) \cup \cdots \cup R_N(w) = W$; w is called a *root* of G.

The following propositions are straightforward consequences of the above definition.

PROPOSITION 29. *Suppose that in the condition of Proposition 27 we also have $\mathsf{G} = \mathsf{G}_1 \sqcup \cdots \sqcup \mathsf{G}_b$ for some rooted N-frames $\mathsf{G}_1, \ldots, \mathsf{G}_b$. Let w_i denote a root of G_i, $i = 1, \ldots, b$. Then for any $w \in \mathsf{M}'$ we have*

$$(\mathsf{M}, w|\overline{\Phi}) \vDash \varphi \Leftrightarrow (\mathsf{M}', w|(\Phi'_1, \ldots, \Psi'_N)) \vDash \varphi,$$

where $\Phi'_i = \Phi_i$ for $i \neq k$, and

$$\Phi'_k = \Phi_k \cup \bigcup_{1 \leq i \leq b} \{\psi \in Sub(\varphi) \mid (\mathsf{M}, w_i|\overline{\Phi}) \vDash \psi \vee \Diamond_1 \psi \vee \cdots \vee \Diamond_N \psi\}.$$

PROPOSITION 30. *Let F be a class of rooted N-frames, $1 \leq k \leq N$, $\mathsf{G} \in \mathcal{PO}$. If G is rooted, $\mathsf{H} \in \mathsf{G}[k; \mathcal{F}]$, then H is rooted.*

DEFINITION 31. As well as in the monomodal case, we say that an N-formula φ is F-*satisfiable under the condition* $\overline{\Psi} = (\Psi_1, \ldots, \Psi_N)$, if φ is true at some root of F in some model over F under the condition $\overline{\Psi}$.

For an N-formula φ, put

$$Sub^*(\varphi) = Sub(\varphi) \cup \{\Diamond_i \psi \mid 1 \leq i, j \leq N, \quad \Diamond_j \psi \in Sub(\varphi)\}.$$

Consider an N-formula φ with $Sub^*(\varphi)_\lessdot = (\psi_1, \ldots, \psi_n)$. For vectors $\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_N \in \{0,1\}^n$, the notation

$$\mathsf{F} \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}$$

means that $Sub^*(\varphi)^{\mathbf{v}}$ is F-satisfiable under the condition

$$(Sub^*(\varphi)_{\mathbf{u}_1}, \ldots, Sub^*(\varphi)_{\mathbf{u}_N}).$$

(Note that if $N = 1$ then $Sub^*(\varphi) = Sub(\varphi)$, so this notation does not contradict the monomodal case). For a class $\mathcal{F}$ of rooted N-frames, $\mathcal{F} \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}$ means that $\mathsf{F} \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}$ for some $\mathsf{F} \in \mathcal{F}$.

Also, for $1 \leq k \leq N$, we define auxiliary function f_k. For boolean vector $\mathbf{v} = (v_1, \ldots, v_n)$, we put $f_k(\mathbf{v}) = (v'_1, \ldots, v'_n)$, where $v'_i = 1$ iff $v_i = 1$ or for some j, l, k' we have $\psi_i = \Diamond_k \psi_l$, $\psi_j = \Diamond_{k'} \psi_l$, and $v_j = 1$.

DEFINITION 32. For a positive integer d, a sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of sets of rooted N-frames is called d-*moderate*, if there exists an algorithm such that for any N-formula φ, for any vectors $\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_N \in \{0,1\}^{|Sub^*(\varphi)|}$, it decides whether

$$\mathcal{F}_{|Sub^*(\varphi)|} \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}$$

Table 2. Algorithm $\mathrm{SatTree_N}$

Function $\mathrm{SatTree_N}$(*formula* φ; *boolean*
vectors $\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_N$; *integers* h, b, k) returns boolean;

```
Begin
 if SatModerate_N(φ, v, u_1, ..., u_N) then return(true);
 if h > 1 then
   for every integer b' such that 1 ≤ b' ≤ b
     for every boolean vectors v_1, ..., v_b' ∈ {0,1}^⟨φ⟩
       if SatModerate_N(φ, v, u_1, ..., u_k ∨ f_k(v_1) ··· ∨ f_k(v_b'), ..., u'_N) then
         if ⋀_{1≤j≤b'} SatTree_N(φ, v_j, u_1, ..., u_N, h − 1, b, k) then
           return(true);
 return(false);
End.
```

in space $O(|Sub^*(\varphi)|^d)$.

The following statement is a straightforward generalization of Proposition 19.

PROPOSITION 33. *Let $\mathcal{F}$ ba a class or rooted N-frames, φ be an N-formula, $n = |Sub^*(\varphi)|$, $1 \le k \le N$. Then for any $\mathbf{u}_1, \ldots, \mathbf{u}_N, \mathbf{v} \in \{0,1\}^n$, for any integers $h, b \ge 1$, the following two conditions are equivalent.*

1. $\mathcal{T}_{h+1,b}[k;\mathcal{F}] \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}$.
2. *Either $\mathcal{F} \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}$ or there exist vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{b'} \in \{0,1\}^n$, where $1 \le b' \le b$, such that*

$$\mathcal{T}_{h,b}[k;\mathcal{F}] \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}_j \text{ for all } j = 1, \ldots, b', \text{ and}$$

$$\mathcal{F} \mid (\mathbf{u}'_1, \ldots, \mathbf{u}'_N) \Vdash_\varphi \mathbf{v},$$

where $\mathbf{u}'_i = \mathbf{u}_i$ for $i \ne k$, $\mathbf{u}'_k = \mathbf{u}_k \vee f_k(\mathbf{v}_1) \cdots \vee f_k(\mathbf{v}_{b'})$.

Proof. By Propositions 29 and 30, analogously to the proof of Proposition 19. ■

COROLLARY 34. *Let $\mathcal{F}$ be a class of rooted N-frames. Suppose that* $\mathrm{SatModerate_N}$ *is an algorithm such that for any N-formula φ, for any $\mathbf{u}_1, \ldots, \mathbf{u}_N, \mathbf{v} \in \{0,1\}^{|Sub^*(\varphi)|}$, it decides whether*

$$\mathcal{F} \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}.$$

Then $\mathrm{SatTree_N}$ *(see Table 2) is an algorithm such that for any formula φ, for any $\mathbf{u}_1, \ldots, \mathbf{u}_N, \mathbf{v} \in \{0,1\}^{|Sub^*(\varphi)|}$, for any integers $h, b \ge 1$, it decides whether*

$$\mathcal{T}_{h,b}[k;\mathcal{F}] \mid (\mathbf{u}_1, \ldots, \mathbf{u}_N) \Vdash_\varphi \mathbf{v}.$$

Since at every step of recursion the algorithm $\mathrm{SatTree_N}$ uses $O(n^2)$ amount of memory for a formula φ with $|Sub^*(\varphi)| = n$, we obtain

THEOREM 35. *If $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is d-moderate sequence of sets of rooted N-frames, $1 \le k \le N$, P is a polynomial of degree d', then the sequence*

$$(\mathcal{T}_{P(n),P(n)}[k; \mathcal{F}_n])_{n\in\mathbb{N}}$$

is $\max\{2 + d', d\}$*-moderate.*

5 PSPACE-decidability of GLP

Is this section we construct PSPACE-decision procedure for GLP.

First, we need to quote some results from [2].

For an N-frame $\mathsf{F} = (W, R_1, \dots, R_N)$ let F_+ denote the $(N + 1)$-frame $(W, \varnothing, R_1, \dots, R_N)$, and let F_∞ denote the frame $(W, R_1, \dots, R_N, \varnothing, \varnothing, \dots)$ with countably many relations.

DEFINITION 36 ([2]). For $N \ge 1$, we inductively define a class $\mathcal{F}^{(N)}$ of N-frames. Let $\mathcal{F}^{(1)}$ be the class of all finite strict partial orders,

$$\mathcal{F}^{(N+1)} = \mathcal{PO}[1; \mathcal{G}^{(N)}], \text{ where } \mathcal{G}^N = \{\mathsf{F}_+ \mid \mathsf{F} \in \mathcal{F}^{(N)}\}.$$

Also put $\mathcal{F}^{\mathrm{J}}=\{\mathsf{F}_\infty \mid \mathsf{F} \in \mathcal{F}^{(N)} \text{ for some } N\}$.

Let J be the logic of the class $\mathcal{F}^{\mathrm{J}}$ (complete axiomatization of this logic is given in [2]). These frames are called *hereditary strict orders*, and were introduced in [2], where the following result was proved:

THEOREM 37 ([2]). *There exists a polynomial-time translation f such that for any formula φ we have*

$$\mathrm{GLP} \vdash \varphi \Leftrightarrow \mathrm{J} \vdash f(\varphi).$$

PROPOSITION 38 ([2]).
(1) If $(W, R_1, \dots, R_N, R, S_1, \dots, S_K) \in \mathcal{F}^{(N+K+1)}$ then $(W, R_1, \dots, R_N, S_1, \dots, S_K)$ is isomorphic to some frame from $\mathcal{F}^{(N+K)}$.
(2) If $(W, R_1, \dots, R_N, S_1, \dots, S_K) \in \mathcal{F}^{(N+K)}$ then $(W, R_1, \dots, R_N, \varnothing, S_1, \dots, S_K)$ is isomorphic to some frame from $\mathcal{F}^{(N+K+1)}$.

Proof. The proof is straightforward (by Definition 36).

Another proof is based on the first-order conditions that characterized the class of hereditary strict orders, see [2] for details. ■

LEMMA 39. *Let φ be an N-formula, and let $\{\Diamond_{i_1}, \dots, \Diamond_{i_K}\}$ be the set of all diamonds that occur in φ, where $i_1 < i_2 < \dots < i_K$. Let φ' be the K-modal formula that obtained from φ by replacing $\Diamond_{i_j}$ with $\Diamond_j$ for all $j = 1 \dots K$. Then φ is $\mathcal{F}^{(N)}$-satisfiable iff φ' in $\mathcal{F}^{(K)}$-satisfiable.*

Proof. $\Rightarrow$). Suppose that φ is satisfiable in a frame $\mathsf{F} = (W, R_1, \ldots, R_N) \in \mathcal{F}^{(N)}$. Clearly, φ' is satisfiable at the frame $\mathsf{G} = (W, R_{i_1}, \ldots, R_{i_K})$. G is obtained from F by 'deleting' some relations, so by Proposition 38, an isomorphic copy of G belongs to $\mathcal{F}^{(K)}$. Thus φ' is $\mathcal{F}^{(K)}$-satisfiable.

$\Leftarrow$). Suppose that φ' is satisfiable in a frame $\mathsf{G} = (W, S_1, \ldots, S_K) \in \mathcal{F}^{(K)}$. Then φ is satisfiable in $\mathsf{F} = (W, R_1, \ldots, R_N)$, where $R_{i_j} = S_j$ for all $j = 1 \ldots K$, and all other relations of F are empty. By Proposition 38, φ is $\mathcal{F}^{(N)}$-satisfiable. ■

For $N, h, b \geq 1$, by induction on N we define a class $\mathcal{T}_{h,b}^{(N)}$. Let $\mathcal{T}_{h,b}^{(1)} = \mathcal{T}_{h,b}[\{\mathsf{C}_0\}]$, i.e., $\mathcal{T}_{h,b}^{(1)}$ is the class (up to isomorphisms) of all finite transitive irreflexive trees with the height not more then h and the branching not more then b. Put

$$\mathcal{T}_{h,b}^{(N+1)} = \mathcal{T}_{h,b}[1; \{\mathsf{F}_+ \mid \mathsf{F} \in \mathcal{T}_{h,b}^{(N)}\}].$$

THEOREM 40. *The satisfiability problem for* J *is in* PSPACE.

Proof. Consider an N-formula φ. Let $n = \langle \varphi \rangle$. By Lemma 39, we may assume that $N < n$.

Suppose that φ is J-satisfiable. Then φ is satisfiable at some N-frame $\mathsf{F} \in \mathcal{F}^{(N)}$. Using Lemma 12, by induction on N, one can show that φ is satisfiable at the root of some N-frame $\mathsf{T} \in \mathcal{T}_{n,n}^{(N)}$. Thus

$$\varphi \text{ is J-satisfiable} \iff \varphi \text{ is } \mathcal{T}_{n,n}^{(N)}\text{-satisfiable}.$$

By induction on N we obtain that there exists d, such that for any N the sequence $(\mathcal{T}_{n,n}^{(N)})_{n \in \mathbb{N}}$ is d-moderate (Theorem 35), and, moreover, we obtain that it is possible to check whether φ is $\mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}^{(\langle\varphi\rangle)}$-satisfiable in polynomial of $\langle \varphi \rangle$ space. Thus satisfiability problem for J is in PSPACE. ■

THEOREM 41. GLP *is in* PSPACE.

Proof. Follows from Theorems 37 and 40. ■

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Ilya Shapirovsky
Institute for Information Transmission Problems
Russian Academy of Sciences
B.Karetny 19, Moscow, Russia, 127994
shapir@iitp.ru