Absolute Completeness of $\mathsf{S4}_u$ for Its Measure-Theoretic Semantics

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Abstract

Given a measure space $\langle X, \mu \rangle$, we define its *measure algebra* $\mathbb{A}_\mu$ as the quotient of the algebra of all measurable subsets of X modulo the relation $X \overset{\mu}{\sim} Y$ if $\mu(X \triangle Y) = 0$. If further X is endowed with a topology $\mathcal{T}$, we can define an interior operator on $\mathbb{A}_\mu$ analogous to the interior operator on $\mathcal{P}(X)$. Formulas of $\mathsf{S4}_u$ (the modal logic $\mathsf{S4}$ with a universal modality $\forall$ added) can then be assigned elements of $\mathbb{A}_\mu$ by interpreting $\Box$ as the aforementioned interior operator.
In this paper we prove a general completeness result which implies the following two facts:

(i) the logic $\mathsf{S4}_u$ is complete for interpretations on any subset of Euclidean space of positive Lebesgue measure;

(ii) the logic $\mathsf{S4}_u$ is complete for interpretations on the Cantor set equipped with its appropriate fractal measure.

Further, our result implies in both cases that given $\varepsilon > 0$, a satisfiable formula can be satisfied everywhere except in a region of measure at most ε.

Keywords: Modal logic, topological semantics, measure theory

1 Introduction

One of the primary appeals of modal logic is the flexibility in its interpretation. Since $\Box$ could be taken to have many different meanings, the same modal logic can often be used in several seemingly unrelated contexts.

The logic $\mathsf{S4}$ is a particularly good example of this, because along with its relational many-worlds semantics, it can be given a topological interpretation, as was already known by McKinsey and Tarski before 1940. With these semantics, modal logic can be used for reasoning about space, a perspective which has proven to be very fruitful. [1] Perhaps the most famous theorem in this field is McKinsey and Tarski's result that $\mathsf{S4}$ is complete for topological interpretations on the real line and, more generally, for any separable metric space without isolated points [11]. More recently, this result has been followed by new proofs and strengthenings for the real line [4,13], as well as the Cantor set [12].

The result is modified slightly when we consider the universal modality from [9], giving rise to the logic $\mathsf{S4}_u$. We once again have completeness of $\mathsf{S4}_u$ for the class of finite topological spaces, but in general these must be disconnected [3]. In Euclidean spaces (and any other connected, separable metric spaces without isolated points) [14] proves that the logic we obtain is $\mathsf{S4}_u + \mathrm{Conn}$, where Conn denotes the connectedness axiom $\forall(\Box p \vee \Box q) \to \exists(\Box p \wedge \Box q)$.

This shows that a well-understood and -behaved modal logic can be used without trouble to reason about topological spaces, despite their richness and complexity. But why stop at topology? We can interpret $\mathsf{S4}$ over spaces which have even deeper structure. The real line, for example, about which much work on $\mathsf{S4}$ has focused, admits not only a natural metric (which is used to interpret the modal operator $\Box$) but also a natural measure. Thus in addition to the question *Can we satisfy a given formula φ on a model based on the real line?* we can ask *Can we satisfy a formula φ with a high probability on a model based on the real line?*

Formulas of $\mathsf{S4}$ can be interpreted over subsets of Euclidean space "up to measure zero"; that is, over the algebra of measurable sets modulo all null sets. This intepretation was called to my attention in a lecture given by Dana Scott in the conference Topology, Algebra and Categories in Logic, 2009. I immediately became interested in the question of finding an analogue to McKinsey and Tarski's theorem.

Here we should remark that topological completeness of $\mathsf{S4}$ does not *a priori* imply its measure-theoretic completeness, or vice-versa. It is true that every model of $\mathsf{S4}$ based on Euclidean space gives rise to a measure-theoretic model (provided that all valuations of propositional variables are measurable); simply take the original valuation modulo null sets. However, the resulting model does not satisfy the same set of formulae. Indeed, many sets which are topologically "large", such as the set of rational numbers which is dense in the real line (or even a dense G_δ, which is topologically large in a more precise sense) can have measure zero and hence "disappear" under our measure-theoretic interpretation. Because of this, even a formula that was topologically satisfied by every point may no longer be satisfied after doing away with null sets.

As an example, consider the formula $\forall(\Diamond p \wedge \Diamond\neg p)$. This formula can be satisfied topologically on the real line by interpreting p as the set of rational numbers. Since both the interpretation of p and its complement are dense, it follows that every point satisfies $\Diamond p \wedge \Diamond\neg p$ and hence $\forall(\Diamond p \wedge \Diamond\neg p)$.

Meanwhile, if we were to translate this directly into a measure-theoretic model, we would be interpreting p as a null set because the set of rationals has measure zero.

[1] Although the basic modal language is not too expressive over the class of topological spaces, there are polymodal systems which turn out to be surprisingly powerful, such as the polymodal Gödel-Löb logic GLP [2] and Dynamic Topological Logic [1,10].

Therefore, every point would satisfy $\neg\Diamond p$, and our original formula would be false everywhere.

In order to give a measure-theoretic model, we would need to interpret p as a set such that every open set U intersects both p and its complement with positive measure. Such a set exists, but the reader unfamiliar with how to construct it may find doing so quite challenging! We will not give an explicit solution, but one can be extracted from our more general completeness proof.

Along these lines, existing proofs of topological completeness cannot serve as proofs of measure-theoretic completeness simply because it is not clear which of the sets that are generated have positive measure and which do not. This of course does not rule out modifying these proofs, taking care of the technical issues arising in the measure-theoretic setting: precisely what we shall set out to do.

On the other hand, working with measure-theoretic semantics has some advantages which might inspire us to think that S4 and related systems are sometimes more likely to be measure-theoretically complete than topologically complete. The reason for this is that there are several extensions of S4 which are incomplete for topological interpretations on Euclidean spaces precisely because said spaces are topologically connected; two examples of this are $\mathsf{S4}_u$, as mentioned above, and *Dynamic Topological Logic*, which can be shown to be incomplete for the plane due to local connectedness [2] [7]. However, measure-theoretically, Euclidean space is quite disconnected. Recall that a topological space is disconnected if it contains proper subsets which are both open and closed. Well, open balls in Euclidean space are both open and closed up to measure zero, because their boundaries carry no measure.

It is the author's opinion that there should be many more measure-theoretic completeness results to be found where topological completeness fails, but here we shall limit our discussion to $\mathsf{S4}_u$. Our main results are that $\mathsf{S4}_u$ is complete for interpretations on the measure algebra of any subset of $\mathbb{R}^N$ which has positive measure (the real line and the unit interval are examples of this, but this class of sets is much more general) and for interpretations on the measure algebra of the Cantor set, where we must take the Hausdorff measure of appropriate fractal dimension (in this case, $\ln(2)/\ln(3)$; see Appendix A). Further, in all of the above cases, if we take any $\varepsilon > 0$, a satisfiable formula φ can be satisfied everywhere except for a set of measure at most ε; in the case that the set we began with was a probability space (such as the unit interval), this means that every satisfiable formula can be satisfied with probability arbitrarily close to one.

2 Syntax and semantics

We will work in a bimodal language $\mathcal{L}$ consisting of propositional variables $p \in PV$ with the Boolean connectives $\neg$ and $\wedge$ (other Booleans are defined in the standard way) and two modal operators, $\Box$ and $\forall$.

[2] Dynamic Topological Logic is also incomplete for the real line but this can be shown using a formula which is not valid on all locally connected spaces [15].

The logic $\mathsf{S4}_u$ is that obtained by all $\mathsf{S4}$-axioms for $\Box$:

$$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi),$$
$$\Box\varphi \rightarrow \varphi,$$
$$\Box\varphi \rightarrow \Box\Box\varphi;$$

all **S5**-axioms for $\forall$ ($\mathsf{S4}$ with the additional axiom $\exists\varphi \rightarrow \forall\exists\varphi$) and the 'bridge' axiom $\forall\varphi \rightarrow \Box\varphi$, together with propositional tautologies, necessitation for both operators and modus ponens.

We wish to define semantics for $\mathsf{S4}_u$ on topological measure spaces, which we define below[3] :

Definition 2.1 [Measure algebra] Let $\mathfrak{X} = \langle X, \mathcal{A}, \mu \rangle$ be a measure space. We define the *measure algebra* of $\mathfrak{X}$, which we will denote $\mathbb{A}_\mu$, to be the set of equivalence classes of $\mathcal{A}$ under the relation $\overset{\mu}{\sim}$ given by $E \overset{\mu}{\sim} F$ if and only if $\mu(E \triangle F) = 0$.

In this paper we will refer to elements of $\mathbb{A}_\mu$ as *regions*.

Denote the equivalence class of $S \in \mathcal{A}$ by $[S]_\mu$. Boolean operations can be defined on $\mathbb{A}_\mu$ in the obvious way; $[E]_\mu \sqcap [F]_\mu = [E \cap F]_\mu$, $[E]_\mu - [F]_\mu = [E \setminus F]_\mu$. We can also define $[E]_\mu \sqsubseteq [F]_\mu$ by $\mu(E \setminus F) = 0$. In general we will use 'square' symbols for notation of the measure algebra and 'round' symbols for set notation in order to avoid confusion. As a slight abuse of notation, if $o \in \mathbb{A}_\mu$ and $o = [S]_\mu$ we may write $\mu(o)$ instead of $\mu(S)$; note that this is well-defined, independently of our choice of $S \in o$.

In order to interpret our modal operators, we need to consider measure spaces which also have a topological structure:

Definition 2.2 [topological measure space] A *topological measure space* is a triple $\langle X, \mathcal{T}, \mu \rangle$ where X is a set, $\mathcal{T}$ a topology on X and μ a σ-finite measure such that every open set is μ-measurable.

A set $S \subseteq X$ is *almost open* if $S \overset{\mu}{\sim} U$ for some $U \in \mathcal{T}$. The region $[S]_\mu$ is *open* if S is almost open.

Equivalently, we can say $o \in \mathbb{A}_\mu$ is open if $o = [U]_\mu$ for some open set U.

Given a σ-finite measure space $\langle X, \mu \rangle$ and $\mathcal{O} \subseteq \mathbb{A}_\mu$, the supremum of $\mathcal{O}$, which we will denote $\bigsqcup \mathcal{O}$, always exists; see Appendix B for details. With this operation we can define an interior operator on any measure algebra:

Definition 2.3 [interior] Let $\langle X, \mathcal{T}, \mu \rangle$ be a topological measure space and $o \in \mathbb{A}_\mu$. We define the *interior* of o by $o^\Box = \bigsqcup \{[U]_\mu \sqsubseteq o : U \in \mathcal{T}\}$.

Proposition 2.4 *If $\langle X, \mathcal{T}, \mu \rangle$ is a topological measure space and $o \in \mathbb{A}_\mu$,*

(i) *$o^\Box$ is open,*

(ii) *$o^\Box \sqsubseteq o$,*

[3] For a brief review of measure spaces, see Appendix A.

(iii) $(o^{\Box})^{\Box} = o^{\Box}$.

Proof. See Appendix B. □

We are now ready to define our semantics:

Definition 2.5 [Measure-theoretic semantics] If $\langle X, \mathcal{A}, \mu \rangle$ is a topological measure space, a *measurable valuation* on X is a function $[\![\cdot]\!] : \mathcal{L} \to \mathbb{A}_\mu$ satisfying

$$\begin{aligned} [\![\alpha \wedge \beta]\!] &= [\![\alpha]\!] \sqcap [\![\beta]\!] \\ [\![\neg\alpha]\!] &= [X]_\mu - [\![\alpha]\!] \\ [\![\Box\alpha]\!] &= [\![\alpha]\!]^{\Box} \\ [\![\forall\alpha]\!] &= \begin{cases} [X]_\mu & \text{if } [\![\alpha]\!] = [X]_\mu \\ [\varnothing]_\mu & \text{otherwise.} \end{cases} \end{aligned}$$

A *topological measure model* is a topological measure space equipped with a measurable valuation.

The system $\mathsf{S4}_u$ is sound for our semantics:

Theorem 2.6 (soundness) *Let $\langle X, \mathcal{T}, \mu, [\![\cdot]\!] \rangle$ be a topological measure model. Then, for every formula φ which is derivable in $\mathsf{S4}_u$, $[\![\varphi]\!] = [X]_\mu$.*

Proof. This follows from the fact that all axioms are valid and all rules preserve validity; note that the $\mathsf{S4}$ axioms for $\Box$ are a direct consequence of Proposition 2.4. □

3 μ-Bisimulations

Our completeness proof depends on a well-known result that $\mathsf{S4}_u$ is complete for the class of finite Kripke frames where the accessibility relation is a preorder (that is, reflexive and transitive).

Definition 3.1 [Kripke frame; Kripke model] A (transitive, reflexive) *Kripke frame* is a preordered set $\langle W, \preccurlyeq \rangle$.

A *Kripke model* is a Kripke frame equipped with a valuation $(\!|\cdot|\!) : \mathcal{L} \to 2^W$ satisfying the standard clauses for Boolean operators,

$$w \in (\!|\Box\varphi|\!) \Leftrightarrow \forall v \preccurlyeq w, v \in (\!|\varphi|\!)$$

and

$$w \in (\!|\forall\varphi|\!) \Leftrightarrow \forall v \in W, v \in (\!|\varphi|\!).$$

The following well-known result can be found, for example, in [3]:

Theorem 3.2 *$\mathsf{S4}_u$ is complete with respect to the class of all finite, transitive, reflexive Kripke models.*

In order to prove our main result, we shall construct a type of bisimulation between a topological measure space and a given Kripke frame. For this we need to define the proper notion of bisimulation. In what follows, $\downarrow w = \{v : v \preccurlyeq w\}$ and a set $U \subseteq W$ is *open* if, for all $w \in U$, $\downarrow w \subseteq U$.

Definition 3.3 [almost continuous, strongly open] Let $\langle X, \mathcal{T}, \mu, [\![\cdot]\!]\rangle$ be a topological measure model and $\langle W, \preccurlyeq, (\!|\cdot|\!)\rangle$ a Kripke model.

Given a partial function[4] $\beta : X \to W$ and $S \subseteq X$, define $\beta[S]_\mu$ to be the set of all $w \in W$ such that $\beta^{-1}(w) \cap S$ has positive measure.

A partial function $\beta : X \to W$ is *almost continuous* if $\beta^{-1}(\downarrow w)$ is almost open for all $w \in W$. It is *strongly open* if whenever S is almost open, $\beta[S]_\mu$ is open, and *strongly surjective* if $\beta^{-1}(w)$ has positive measure for all $w \in W$.

Definition 3.4 [μ-Bisimulation]

With notation as above, a μ-*bisimulation* is a partial function $\beta : X \to W$ which is

(i) almost continuous,
(ii) strongly open,
(iii) defined almost everywhere,
(iv) strongly surjective and
(v) satisfies $[\![p]\!] = [\beta^{-1}(\!|p|\!)]_\mu$ for all $p \in PV$.

μ-Bisimulations preserve valuations of formulae. Before proving this fact we need a preliminary lemma.

Lemma 3.5 *If $\langle X, \mu\rangle$ is a measure space, W a finite set and $\beta : X \to W$ a partial function defined almost everywhere, then for every measurable $S \subseteq X$, $[S]_\mu \sqsubseteq [\beta^{-1}\beta[S]_\mu]_\mu$.*

Proof. Clearly

$$[S]_\mu = \bigsqcup_{w\in W} \left([\beta^{-1}(w)]_\mu \sqcap [S]_\mu\right),$$

since β is defined almost everywhere and W is finite. Now,

$$[\beta^{-1}(w)]_\mu \sqcap [S]_\mu = [\varnothing]_\mu$$

unless $w \in \beta[S]_\mu$, so we can write

$$\begin{aligned}
\textstyle\bigsqcup_{w\in W} \left([\beta^{-1}(w)]_\mu \sqcap [S]_\mu\right) &= \textstyle\bigsqcup_{w\in\beta[S]_\mu} [\beta^{-1}(w)]_\mu \sqcap [S]_\mu \\
\text{(Lemma B.2)} \qquad &= \left[\textstyle\bigcup_{w\in\beta[S]_\mu} \left(\beta^{-1}(w) \cap S\right)\right]_\mu \\
&\sqsubseteq \left[\textstyle\bigcup_{w\in\beta[S]_\mu} \left(\beta^{-1}(w)\right)\right]_\mu \\
&= \left[\beta^{-1}\beta[S]_\mu\right].
\end{aligned}$$

□

[4] That is, a function whose domain is a subset of X and possibly all of X.

Theorem 3.6 *Suppose that* $\langle X, \mathcal{T}, \mu, [\![\cdot]\!]\rangle$ *is a topological measure model,* $\langle W, \preccurlyeq, (\!|\cdot|\!)\rangle$ *a finite Kripke model and* $\beta : X \to W$ *a* μ*-bisimulation. Then, for every formula* φ*,* $[\![\varphi]\!] = [\beta^{-1}(\!|\varphi|\!)]_\mu$.

Proof. The proof follows by a simple induction on the build of formulas, with only the case for $\Box\varphi$ and $\forall\varphi$ being non-standard.

For $\Box\varphi$, note that $(\!|\Box\varphi|\!) \subseteq (\!|\varphi|\!)$ and by induction hypothesis $[\![\varphi]\!] = [\beta^{-1}(\!|\varphi|\!)]_\mu$.

Now, $(\!|\Box\varphi|\!)$ is open in W and since β is almost continuous,

$$[\beta^{-1}(\!|\Box\varphi|\!)]_\mu = \left[\bigcup_{w\in(\!|\Box\varphi|\!)} \beta^{-1}(\downarrow w)\right]_\mu$$

is open, because each $\beta^{-1}(\downarrow w)$ is almost open. But

$$\left[\beta^{-1}(\!|\Box\varphi|\!)\right]_\mu \sqsubseteq [\beta^{-1}(\!|\varphi|\!)]_\mu = [\![\varphi]\!],$$

so $[\beta^{-1}(\!|\Box\varphi|\!)]_\mu \sqsubseteq [\![\varphi]\!]^\Box$ (recall that $[\![\varphi]\!]^\Box$ is the supremum over all open $o \sqsubseteq [\![\varphi]\!]$) and hence $[\beta^{-1}(\!|\Box\varphi|\!)]_\mu \sqsubseteq [\![\Box\varphi]\!]$.

For the other direction, consider $[\![\Box\varphi]\!]$. This is an open region and hence if $w \in \beta[\![\Box\varphi]\!]$, it follows that $\downarrow w \subseteq \beta[\![\Box\varphi]\!]$, because β is strongly open. But $\beta[\![\Box\varphi]\!] \subseteq \beta[\![\varphi]\!]$ and from our induction hypothesis we can see that $\beta[\![\varphi]\!] \subseteq (\!|\varphi|\!)$, so $\downarrow w \subseteq (\!|\varphi|\!)$. This implies that $w \in (\!|\Box\varphi|\!)$ and, given that w was arbitrary, $\beta[\![\Box\varphi]\!] \subseteq (\!|\Box\varphi|\!)$. Applying β^{-1} to both sides and using Lemma 3.5 we conclude that $[\![\Box\varphi]\!] \sqsubseteq [\beta^{-1}(\!|\Box\varphi|\!)]_\mu$, as desired.

The case of $\forall\varphi$ is simpler and uses the fact that β is strongly surjective and defined almost everywhere; we will skip the details. □

4 Provinces

We will focus much of our discussion on what we shall call *provinces*; these are an abstract class of spaces which have the basic properties we need of bounded subsets of Euclidean space, but are more general and include other familiar spaces (such as the Cantor set with its appropriate Hausdorff measure).

Definition 4.1 [Province] A *province* is a triple $\langle X, d, \mu\rangle$ where X is a set, d a metric and μ a measure on X satisfying

(i) every open set is μ-measurable;

(ii) every non-empty open set has finite positive measure;

(iii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that, given $x \in X$, $\mu(B_\delta(x)) < \varepsilon$;

(iv) the boundary of every open ball has measure zero;

(v) X is totally bounded.

Lemma 4.2 *If i and $j_0, ..., j_{N-1}$ are open balls in a province $\langle X, d, \mu\rangle$, then*

$$i \setminus \bigcup_{n<N} j_n \overset{\mu}{\sim} \left(i \setminus \bigcup_{n<N} j_n \right)^{\circ}.$$

Proof. For every $n < N$, $\mu\left(\overline{j}_n \setminus j_n\right) = 0$ because the boundaries of open balls have measure zero in a province. Hence $\overline{j} \overset{\mu}{\sim} j$ and

$$i \setminus \bigcup_{n<N} j_n \overset{\mu}{\sim} i \setminus \bigcup_{n<N} \overline{j}_n.$$

But $\bigcup_{n<N} \overline{j}_n$ is closed (it is a finite union of closed sets) so $i \setminus \bigcup_{n<N} \overline{j}_n$ is an open subset of $i \setminus \bigcup_{n<N} j_n$ (in fact, its interior) and the result follows. □

Lemma 4.3 *If $\langle X, d, \mu\rangle$ is a province and $U \subseteq X$ is non-empty and open, then U is infinite.*

Proof. Suppose towards a contradiction that U was a finite, non-empty, open set and $x \in U$. Then, for δ small enough we have that $B_\delta(x) = \{x\}$ (take δ to be less than the distance between x and any point in U). Now, $B_\delta(x)$ is non-empty and open, so by the definition of province it has positive measure. Then there must exist $\varepsilon \in (0, \delta)$ such that $\mu(B_\varepsilon(x)) < \frac{\mu(B_\delta(x))}{2}$ (once again by the definition of province); but this cannot be, since $B_\varepsilon(x) = B_\delta(x) = \{x\}$, so they must have the same measure. Thus we have reached a contradiction and conclude that U must be infinite. □

5 Constructing μ-bisimulations

In this section we will construct continuous, open maps from an arbitrary province to a preorder. This, along with the Bisimulation Theorem, will establish the completeness of $\mathsf{S4}_u$ for these spaces and, more generally, for countable unions of provinces.

Definition 5.1 [Graded partition] Let $\langle X, d, \mu\rangle$ be a province. A *graded partition*[5] on X is a set $\mathfrak{p}$ of open balls in X such that

(i) $X \in \mathfrak{p}$ and

(ii) if $i, j \in \mathfrak{p}$ and $i \cap j \neq \varnothing$, then either $i \subseteq j$ or $j \subseteq i$.

Definition 5.2 [Placement function] Let $\mathfrak{p}$ be a graded partition on a province $\langle X, d, \mu\rangle$. The *placement function* of $\mathfrak{p}$ is the (possibly partial) function $\mathrm{pl}_\mathfrak{p} : X \to \mathfrak{p}$ assigning to each $x \in I$ the least $i \in \mathfrak{p}$ such that $x \in i$.

We will write $\mathrm{gr}_\mathfrak{p}$ instead of $\mathrm{pl}_\mathfrak{p}^{-1}$ and call $\mathrm{gr}_\mathfrak{p}(i)$ the *ground* of i.

[5] Compare to the *open ball trees* from [7].

Definition 5.3 [Fine graded partition] Let $\eta \in (0,1)$. A graded partition $\mathfrak{p}$ is η-*fine* if for every $i \in \mathfrak{p}$ we have that

$$\mu\left(\bigcup\{j \in \mathfrak{p} : j \subsetneq i\}\right) < \eta \cdot \mu(i).$$

Note that if $\mathfrak{p}$ is an η-fine graded partition and $i \in \mathfrak{p}$, it follows that $\mu(\mathrm{gr}_{\mathfrak{p}}(i)) > 0$; specifically, $\mu(\mathrm{gr}_{\mathfrak{p}}(i)) > (1-\eta)\mu(i)$.

Lemma 5.4 *Let $\eta \in (0,1)$ and $\mathfrak{p}$ be an η-fine graded partition. Then, given any $i \in \mathfrak{p}$, there exist only finitely many $j \in \mathfrak{p}$ such that $i \subseteq j$.*

Proof. Note that if $i \subsetneq j$, then $\mu(i) < \eta \cdot \mu(j)$, because $\mathfrak{p}$ is η-fine.

It follows that if

$$i = i_0 \subsetneq i_1 \subsetneq ... \subsetneq i_n$$

is a chain of elements of $\mathfrak{p}$, then $\mu(i_n) \geq (1/\eta)^n \mu(i)$; since $\mu(X)$ is finite and

$$\lim_{n\to\infty} (1/\eta)^n = \infty,$$

n must be bounded by some fixed $N < \omega$. But the elements of $\mathfrak{p}$ properly containing i are totally ordered, so there can be at most N of these. □

In view of the previous lemma we can give the following definition:

Definition 5.5 [Height] Given an η-fine graded partition $\mathfrak{p}$ and $i \in \mathfrak{p}$, we can define the *height* of i, denoted $\mathrm{hgt}(i)$, as the largest n such that there exists a sequence

$$i = i_0 \subsetneq i_1 \subsetneq ... \subsetneq i_n$$

in $\mathfrak{p}$.

We define $\mathfrak{p}[n]$ to be the set of all elements of $\mathfrak{p}$ of height n.

Lemma 5.6 *Let $\mathfrak{p}$ be an η-fine graded partition on a province $\langle X, d, \mu\rangle$. Then,*

$$\mu\left(\bigcup \mathfrak{p}[n]\right) < \eta^n \mu(X).$$

Proof. We prove this by induction on n. Clearly, every $i \in \mathfrak{p}[n+1]$ is contained in a unique $j \in \mathfrak{p}[n]$; since $\mathfrak{p}$ is η-fine, it follows that, for a fixed $j \in \mathfrak{p}[n]$,

$$\mu\left(\bigcup\{i \in \mathfrak{p}[n+1] : i \in j\}\right) < \eta \cdot \mu(j).$$

Now, writing

$$\mathfrak{p}[n+1] = \bigcup_{j \in \mathfrak{p}[n]} \{i \in \mathfrak{p}[n+1] : i \subseteq j\}$$

we have that

$$\begin{aligned}\mu(\mathfrak{p}[n+1]) &= \mu\left(\bigcup_{j\in\mathfrak{p}[n]}\bigcup\{i\in\mathfrak{p}[n+1]:i\in j\}\right)\\ &= \sum_{j\in\mathfrak{p}[n]}\mu\left(\bigcup\{i\in\mathfrak{p}[n+1]:i\in j\}\right)\\ &< \eta\sum_{j\in\mathfrak{p}[n]}\mu(j)\\ &\overset{\text{IH}}{<} \eta^{n+1}\mu(X),\end{aligned}$$

as desired (note that we are using the fact that elements of $\mathfrak{p}[n]$ are disjoint, a consequence of Definition 5.1.ii; this allows us to commute unions and sums). □

Corollary 5.7 *If $\mathfrak{p}$ is an η-fine graded partition on a province $\langle X, d, \mu\rangle$, then $\mathrm{pl}_{\mathfrak{p}}$ is defined almost everywhere on X.*

Proof. We know that $X \in \mathfrak{p}$ by definition, so $\mathrm{pl}_{\mathfrak{p}}(x)$ can only be undefined if x lies on some $i \in \mathfrak{p}$ but there is no minimal such i. If that is the case, we have an infinite sequence

$$i_0 \supsetneq i_1 \supsetneq \ldots \supsetneq i_n \supsetneq \ldots$$

of elements of $\mathfrak{p}$ containing x; therefore $x \in \bigcup\mathfrak{p}[n]$ for all $n < \omega$.

However, by Lemma 5.6, $\mu\left(\bigcup\mathfrak{p}[n]\right) < \eta^n\mu(X)$, so

$$\mu\left(\bigcap_{n<\omega}\bigcup\mathfrak{p}[n]\right) < \eta^k\mu(X)$$

for all k and must equal zero. Since $\mathrm{pl}_{\mathfrak{p}}$ is defined on the complement of this set in X, it follows that $\mathrm{pl}_{\mathfrak{p}}$ is defined almost everywhere. □

Definition 5.8 [Naming function] Let $\mathfrak{p}$ be a graded partition. A *naming function* on $\mathfrak{p}$ is a function $\nu : \mathfrak{p} \to W$, where W is a preordered set, such that $\nu(i) \preccurlyeq \nu(j)$ whenever $i \subseteq j$.

Given a named graded partition $\langle\mathfrak{p}, \nu\rangle$ on X, we define a partial function $\dot{\nu} : X \to W$ by $\dot{\nu} = \nu \circ \mathrm{pl}_{\mathfrak{p}}$.

Lemma 5.9 *If $\eta \in (0, 1)$ and $\langle\mathfrak{p}, \nu\rangle$ is a named η-fine graded partition on a province $\langle X, d, \mu\rangle$, then $\dot{\nu}$ is defined almost everywhere, almost continuous and, for all $i \in \mathfrak{p}$,*

$$\mu(\{x \in i : \dot{\nu}(x) = \nu(i)\}) > 0.$$

Proof. That $\dot{\nu}$ is defined almost everywhere is an immediate consequence of Lemma 5.7.

To see that $\dot{\nu}$ is continuous, pick $w \in W$ and consider $\downarrow w$. Suppose that $x \in \dot{\nu}^{-1}(w)$, so that $\nu(\mathrm{pl}_{\mathfrak{p}}(x)) = w$. Note that for all $i \in \mathfrak{p}$ such that $i \subsetneq \mathrm{pl}_{\mathfrak{p}}(x)$, we have that $\nu(i) \in \downarrow w$ (by the definition of a naming function). Hence whenever $y \in \mathrm{pl}_{\mathfrak{p}}(x)$ and $\dot{\nu}(y)$ is defined, we have that $\dot{\nu}(y) \in \downarrow w$; since $\dot{\nu}$ is defined almost everywhere, we have that almost every point of $\mathrm{pl}_{\mathfrak{p}}(x)$ lies in $\dot{\nu}^{-1}(\downarrow w)$. Since $\mathrm{pl}_{\mathfrak{p}}(x)$ is an open neighborhood of x we conclude that $\dot{\nu}$ is almost continuous.

The last claim follows from the fact that

$$\mu(\mathrm{gr}_{\mathfrak{p}}(i)) > (1-\eta)\mu(i).$$

□

Definition 5.10 [Refinement; ε-refinement] Let $\langle \mathfrak{p}, \nu \rangle$ and $\langle \mathfrak{p}', \nu' \rangle$ be named graded partitions on a province $\langle X, d, \mu \rangle$. We say $\mathfrak{p}'$ is a *refinement* of $\mathfrak{p}$ if

(i) $\mathfrak{p} \subseteq \mathfrak{p}'$,

(ii) $\nu = \nu' \upharpoonright \mathfrak{p}$ and

(iii) if $i \in \mathfrak{p}$ and $j \in \mathfrak{p}'$ are such that $i \subseteq j$, it follows that $j \in \mathfrak{p}$.

Further, we say $\mathfrak{p}'$ is an *ε-refinement* of $\mathfrak{p}$ if, for almost every $x \in X$ and every $v \preccurlyeq \dot{\nu}(x)$, there exists $i \in \mathfrak{p}'$ such that $i \subseteq B_\varepsilon(x)$ and $\nu'(i) = v$.

Lemma 5.11 *Given $\eta \in (0,1)$ and $\varepsilon > 0$, every finite η-fine graded partition $\mathfrak{p}$ admits a finite ε-refinement which is also η-fine.*

Proof. Let $i \in \mathfrak{p}$, and consider $(\mathrm{gr}_{\mathfrak{p}}(i))^\circ \overset{\mu}{\sim} \mathrm{gr}_{\mathfrak{p}}(i)$ (Lemma 4.2). Because X is totally bounded, there exists a finite set

$$\{x_0, ..., x_{N-1}\} \subseteq (\mathrm{gr}_{\mathfrak{p}}(i))^\circ$$

which is $\varepsilon/2$-dense in $(\mathrm{gr}_{\mathfrak{p}}(i))^\circ$.

Pick $\delta \in (0, \varepsilon)$ such that, for all $x \in X$, $\mu(B_\delta(x)) < \eta/N$, $B_\delta(x_n) \subseteq (\mathrm{gr}_{\mathfrak{p}}(i))^\circ$ whenever $n < N$ and $\delta < d(x_n, x_m)$ whenever $n < m < N$.

Let $M = \# \downarrow \nu(i)$. By Lemma 4.3, $B_\delta(x_n)$ is infinite for each $n < N$, so we can find $y_{0,n}, ..., y_{M,n} \in B_\delta(x_n)$. Then, taking $\iota > 0$ small enough, we can ensure that the balls $B_\iota(y_{m,n})$ are mutually disjoint, contained in $B_\delta(x_n)$, and

$$\mu(B_\iota(x)) < \frac{\eta\mu(i) - \mu(i \setminus \mathrm{gr}_{\mathfrak{p}}(i))}{MN} \tag{1}$$

for all x.

Define $b(i, n, m) = B_\iota(y_{mn})$, $N(i) = N$ and $M(i) = M$. Fix a numbering $w_0^i, ..., w_{M-1}^i$ of the elements of $\downarrow \nu(i)$.

Now, let $\mathfrak{p}' = \mathfrak{p} \cup \{b(i,n,m) : i \in \mathfrak{p}, m < M, n < N(i)\}$ and

$$\nu'(i) = \begin{cases} \nu(i) & \text{if } i \in \mathfrak{p}, \\ \\ w^j_m & \text{if } i = b(j,n,m). \end{cases}$$

It is not hard to see that $\langle \mathfrak{p}', \nu' \rangle$ satisfies the desired properties. In particular, condition 1 guarantees that it remains η-fine. □

Definition 5.12 [Fine sequence of graded partitions] Let $\eta \in (0,1)$. A sequence of named graded partitions $\langle \mathfrak{p}_n, \nu_n \rangle_{n<\omega}$ is *η-fine* if

(i) each $\mathfrak{p}_n$ is η-fine and

(ii) each $\mathfrak{p}_{n+1}$ is an η^n-refinement of $\mathfrak{p}_n$.

We set $\mathfrak{p}_\omega = \bigcup_{n<\omega} \mathfrak{p}_n$ and define $\nu_\omega = \bigcup_{n<\omega} \nu_n$.

Lemma 5.13 *Given any province $\langle X, d, \mu \rangle$, $\eta \in (0,1)$, a preordered set $\langle W, \preccurlyeq \rangle$ and $w \in W$, there exists an η-fine sequence of graded partitions $\langle \mathfrak{p}_n, \nu_n \rangle_{n<\omega}$ with $\mathfrak{p}_0 = \{X\}$ and $\nu_\omega(X) = \nu_0(X) = w$.*

Proof. This follows by applying Lemma 5.11 ω times. □

Lemma 5.14 *If $\langle \mathfrak{p}_n, \nu_n \rangle_{n<\omega}$ is an $\eta/2$-fine sequence of graded partitions, then $\mathfrak{p}_\omega$ is η-fine.*

Proof. Pick $i \in \mathfrak{p}_\omega$, and let N be large enough so that $i \in \mathfrak{p}_N$.

For $N \leq n \leq \omega$ let $S_n = \bigcup \{j \in \mathfrak{p}_n : j \subsetneq i\}$.

Note that $\mu(S_n) < \eta/2 \mu(i)$ whenever $N \leq n \leq \omega$, because $\mathfrak{p}_n$ is η-fine.

Further, $S_n \subseteq S_m$ whenever $n \leq m$ and $S_\omega = \bigcup_{N<n<\omega} S_n$, from which it follows that

$$\mu(S_\omega) = \lim_{n\to\infty} \mu(S_n) \leq \eta/2,$$

the limit holding because μ is a measure.

In particular, $\mu(S_\omega) < \eta$, as desired. □

Lemma 5.15 *If $\langle \mathfrak{p}_n, \nu_n \rangle_{n<\omega}$ is an η-fine sequence of graded partitions, then $\dot{\nu}_\omega$ is strongly open.*

Proof. Let $U \subseteq X$ be a non-empty, open set and $w \in \dot{\nu}_\omega[U]_\mu$. This means that there exists $i \in \mathfrak{p}_\omega$ such that $\nu_\omega(i) = w$ and $\mu(i \cap U) > 0$.

Pick N large enough so that $i \in \mathfrak{p}_N$.

Now, for all $n \geq N$ we have that $\mathrm{gr}_{\mathfrak{p}_n}(i)$ is almost open, hence

$$\mathrm{gr}_{\mathfrak{p}_\omega}(i) = \bigcap_{n \geq N} \mathrm{gr}_{\mathfrak{p}_n}(i)$$

$$\overset{\mu}{\sim} \bigcap_{n \geq N} (\mathrm{gr}_{\mathfrak{p}_n}(i))^\circ;$$

the last step is valid by Lemma 4.2.

Pick $x \in \bigcap_{n \geq N} (\mathrm{gr}_{\mathfrak{p}_n}(i))^\circ$, $\varepsilon > 0$ small enough so that $B_\varepsilon(x) \subseteq U$ and M large enough so that $\eta^M < \varepsilon$.

Then, for every $v \preccurlyeq w$ there exists $j \in \mathfrak{p}_{M+1}$ such that $\nu_{M+1}(j) = v$ and $j \subseteq B_\varepsilon(x)$, because $\mathfrak{p}_{M+1}$ is an η^M-refinement of $\mathfrak{p}_M$.

But $\mu(\mathrm{gr}_{\mathfrak{p}_\omega}(j)) > 0$, and for any $y \in \mathrm{gr}_{\mathfrak{p}_\omega}(j)$ we have $\dot{\nu}_\omega(y) = v$.

Since v was arbitrary, we conclude that $\downarrow w \subseteq \dot{\nu}_\omega[U]_\mu$, and since U was arbitrary $\dot{\nu}_\omega$ is strongly open, as desired. □

Proposition 5.16 *Let $\langle X, d, \mu \rangle$ be a province, $\langle W, \preccurlyeq \rangle$ a finite preordered set and $w \in W$. Then, given $\varepsilon > 0$ there exists an almost continuous, strongly open map*

$$\beta : X \to W$$

such that

$$\mu(\beta^{-1}(W \setminus \{w\})) < \varepsilon.$$

Proof. Let $\eta = \varepsilon/\mu(X)$. By Lemma 5.13, there is an $\eta/2$-fine sequence of named graded partitions $\langle \mathfrak{p}_n, \nu_n \rangle_{n<\omega}$ such that $\nu_\omega(X) = w$. Then we can take $\beta = \dot{\nu}_\omega$.

Since $\mathfrak{p}_\omega$ is η-fine and $\nu_\omega(X) = w$, we have that

$$X \setminus \bigcup \{i \in \mathfrak{p}_\omega : i \subsetneq X\} \subseteq \beta^{-1}(w)$$

and

$$\mu\left(\bigcup \{i \in \mathfrak{p}_\omega : i \subsetneq X\}\right) < \eta \cdot \mu(X).$$

It follows that $\mu(\beta^{-1}(W \setminus \{w\})) < \varepsilon$, as desired. □

Corollary 5.17 *Let $\langle X, d, \mu \rangle$ be a province, $\langle W, \preccurlyeq \rangle$ a finite preordered set and $w \in W$. Then, for every $\varepsilon > 0$ there exists an almost continuous, strongly open surjection $\beta : X \to W$ such that $\mu(\beta^{-1}(W \setminus \{w\})) < \varepsilon$.*

Proof. Let $N = \#W$. By Lemma 4.3, X is infinite so we can find points $x_1, ..., x_{N-1}$ and $\delta > 0$ such that $B_\delta(x_i) \cap B_\delta(x_j) = \varnothing$ whenever $i \neq j$. Taking δ small enough, we can ensure that

$$\mu\left(X \setminus \bigcup_{n<N} B_\delta(x_n)\right) > 0$$

and

$$\mu\left(\bigcup_{n<N} B_\delta(x_n)\right) < \varepsilon/2.$$

Define

$$X_0 = X \setminus \bigcup_{n<N} B_\delta(x_n)$$

and $X_n = B_\delta(x_n)$ for $1 \le n < N$. By Lemma 4.2, X_0 is almost open, as are the rest of the X_n.

Write

$$W = \{w = w_0, w_1, ..., w_{N-1}\}.$$

Clearly each X_n is a province, so by Proposition 5.16 there exist almost continuous, strongly open maps $\beta_n : X_n \to W$ with $\mu(\beta_n^{-1}(w_n)) \neq 0$ and $\mu(X_0 \setminus \beta^{-1}(w)) < \varepsilon/2$. Then we can set $\beta = \bigcup_{n<N} \beta_n$; it is not hard to see that β has all the desired properties. □

These results can be generalized to structures we will call *territories*:

Definition 5.18 [Territory] A *territory* is a triple $\langle X, d, \mu\rangle$ such that $X = \bigcup_{n<N} X_n$, where $N \le \omega$, the sets X_n are disjoint, almost open subsets of X and

$$\langle X_n, d \restriction X_n, \mu \restriction X_n\rangle$$

is a province for all $n < N$.

Note that in the above definition, if N is finite then $\langle X, d, \mu\rangle$ is itself a province.

Corollary 5.19 *Let $\langle X, d, \mu\rangle$ be a territory.*

Then, for every $w \in W$ and $\varepsilon > 0$ there exists a strongly surjective, almost continuous and strongly open map $\beta : X \to W$ such that

$$\mu\left(X \setminus \beta^{-1}(w)\right) < \varepsilon.$$

Proof. Write $X = \bigcup_{n<N} X_n$ as in Definition 5.18. By Corollary 5.17, for all $n < N$ there is a surjective, continuous, open map $\beta_n : X_n \to W$ such that

$$\mu\left(X_n \setminus \beta^{-1}(w)\right) < \varepsilon/2^{n+1},$$

so that

$$\begin{aligned}\mu\left(\textstyle\bigcup_{n<\omega} X_n \setminus \beta^{-1}(w)\right) &= \textstyle\sum_{n<\omega} \mu(X_n \setminus \beta^{-1}(w))\\ &< \textstyle\sum_{n<\omega} \varepsilon/2^{n+1}\\ &= \varepsilon.\end{aligned}$$

We can then take

$$\beta = \bigcup_{n<N} \beta_n.$$

□

6 Completeness results

In this section we will prove completeness of $\mathsf{S4}_u$ for its measure-theoretic semantics on subsets of Euclidean space as well as the Cantor set with its fractal measure. In fact we will use a stronger notion of completeness:

Definition 6.1 [Absolute completeness] Let $\mathfrak{X} = \langle X, \mathcal{T}, \mu \rangle$ be a topological measure space with $\mu(X) > 0$ and φ a formula of $\mathcal{L}$.

We say φ is *absolutely satisfiable* on $\mathfrak{X}$ if, for every $\varepsilon > 0$ there exists a valuation $[\![\cdot]\!]$ on $\mathfrak{X}$ such that $\mu([\![\neg\varphi]\!]) < \varepsilon$.

A logic Λ is *absolutely complete* for $\mathfrak{X}$ if for every $\varphi \in \mathcal{L}$, $\neg\varphi \notin \Lambda$ implies that φ is absolutely satisfiable on $\mathfrak{X}$.

Note that absolute satisfiability implies satisfiability and absolute completeness implies completeness.

Our main results are direct consequences of the following more general theorem:

Theorem 6.2 *Let φ be a formula of $\mathsf{S4}_u$ such that $\neg\varphi \notin \mathsf{S4}_u$. Suppose that $\langle X, d, \mu \rangle$ is a territory.*

Then, φ is absolutely satisfiable on $\langle X, d, \mu \rangle$.

Proof. Suppose φ is a satisfiable formula. Then, by Theorem 3.2 there exists a finite Kripke model $\langle W, \preccurlyeq, (\!|\cdot|\!) \rangle$ with some $w_* \in (\!|\varphi|\!)$.

By Corollary 5.19, there exists an almost continuous, open surjection $\beta : X \to W$ such that $\mu\left(X \setminus \beta^{-1}(w_*)\right) < \varepsilon$.

Setting $[\![\cdot]\!] = \beta^{-1}(\!|\cdot|\!)$, β becomes a μ-bisimulation, and by Theorem 3.4, $[\![\neg\varphi]\!] \sqsubseteq [X \setminus \beta^{-1}(w_*)]_\mu$, from which the result follows. $\square$

As we will see, our main completeness results, for subsets of Euclidean space and the Cantor set, are special cases of Theorem 6.2.

In what follows, $|\cdot|$ denotes the N-dimensional Lebesgue measure.

Lemma 6.3 *Let X be a Lebesgue-measurable subset of $\mathbb{R}^N$ of positive measure. Then, up to a set of measure zero, X is a territory.*

Proof. First assume X is bounded. We will show that it is already a province.

X is totally bounded (as is every bounded subset of Euclidean space), and clearly for every $\varepsilon > 0$ there is $\delta > 0$ such that $|B_\delta(x) \cap X| < \varepsilon$ for all $x \in X$ (use the same δ that works for all of $\mathbb{R}^N$).

Now, let Y be the set of all Lebesgue points of density of X (see Appendix A). $|X \triangle Y| = 0$, and if $U \subseteq Y$ is open in Y and $x \in U$, then for ε small enough we have $B_\varepsilon(x) \cap Y \subseteq U$ and

$$|B_\varepsilon(x) \cap Y| > \frac{|B_\varepsilon(x)|}{2};$$

this implies that $|U| > 0$.

Finally, the boundary of every open ball in Y has measure zero because the boundary of any open ball in $\mathbb{R}^N$ does.

We conclude that Y is a province, and $Y \sim X$, as desired.

Now, if X is not bounded, write

$$X = \bigcup_{\mathbf{k} \in \mathbb{Z}^N} \{\mathbf{x} \in X : x_n \in [k_n, k_{n+1}) \text{ for all } n < N\}.$$

This is a countable, disjoint union of bounded subsets of $\mathbb{R}^N$ and hence each component which has positive measure can be written as a province, up to a set of measure zero. Clearly, X is equivalent to the union of these components. □

Corollary 6.4 *Given any Lebesgue-measurable $X \subseteq \mathbb{R}^N$ of positive measure,* $\mathsf{S4}_u$ *is absolutely complete for X.*

Proof. Immediate from Theorem 6.2 and Lemma 6.3. □

Corollary 6.5 $\mathsf{S4}_u$ *is absolutely complete for the Cantor set under the* $\ln(2)/\ln(3)$-*Hausdorff measure*[6] .

Proof. Immediate from Proposition 5.16 and the fact that the Cantor set is a province under this measure. □

A Notions from measure theory

Here we review some notions from measure theory that are used throughout the text; we will assume basic familiarity with metric and topological spaces. All of the background we need should be covered in any standard text on real analysis and measure theory, such as [8].

If $\langle X, \mathcal{T} \rangle$ is a topological space and $S \subseteq X$, we will use S° to denote the topological interior of S and $\overline{S}$ to denote its closure.

If $\langle X, d \rangle$ is a metric space, $x \in X$ and $\varepsilon > 0$, then $B_\varepsilon(x)$ denotes the open ball around x with radius ε; that is, the set of all $y \in X$ such that $d(x, y) < \varepsilon$. Every metric space natuarlly acquires a topology given by $U \subseteq X$ being open if and only if, whenever $x \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. All metric spaces will be assumed to be endowed with this topology.

A metric space $\langle X, d \rangle$ is *totally bounded* if for all $\varepsilon > 0$ there exist finitely many elements $x_0, ..., x_{N-1} \in X$ such that for every $y \in X$ there is $n < N$ with $d(y, x_n) < \varepsilon$. Every bounded subset of Euclidean space is totally bounded.

A *measure space* is a triple $\langle X, \mathcal{A}, \mu \rangle$ where X is a set, $\mathcal{A} \subseteq 2^X$ is a σ-algebra (that is, a collection of sets containing $\varnothing$ and X which is closed under set difference and countable unions) and $\mu : \mathcal{A} \to [0, \infty]$ (the non-negative reals with a maximal element ∞ added) satisfying

(i) $\mu(\varnothing) = 0$

(ii) $\mu(A \setminus B) = \mu(A) - \mu(B)$ if $B \subseteq A$ and

[6] See Appendix A.

(iii) if $\langle A_n \rangle_{n<\omega}$ is an increasing sequence of elements of $\mathcal{A}$,

$$\mu\left(\bigcup_{n<\omega} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Elements of $\mathcal{A}$ will be called *μ-measurable*. Note that $\mathcal{A}$ can be reconstructed from μ, since it is the domain of μ; because of this we will often omit explicit mention of $\mathcal{A}$ and speak of measure spaces as pairs $\langle X, \mu\rangle$. We say μ is *σ-finite* if there are countably many sets $S_n \subseteq X$ such that $\mu(S_n)$ is finite for all $n < \omega$ and $X = \bigcup_{n<\omega} S_n$. Measure spaces which are σ-finite cannot contain an uncountable collection of disjoint sets of positive measure.

We always assume that Euclidean space $\mathbb{R}^N$ is equipped with the standard Euclidean metric and Lebesgue measure; the latter will be denoted $|\cdot|$.

Given a set $S \subseteq \mathbb{R}^N$ and $x \in \mathbb{R}^N$, x is a *Lebesgue point of density* of S if

$$\lim_{\varepsilon\to 0} \frac{|B_\varepsilon(x) \cap S|}{|B_\varepsilon(x)|} = 1.$$

It is a famous theorem of Lebesgue that for every measurable set $S \subseteq \mathbb{R}^N$, almost every $x \in S$ is a point of density of S; that is, the set of elements of S which are not Lebesgue points of S has measure zero. For a proof and further details see, for example, [5].

The Cantor set has Lebesgue measure zero. However, it has measure one under the $\ln(2)/\ln(3)$-dimensional Hausdorff measure. Hausdorff measures can be used to measure fractals (sets of non-integer dimension) and provide a generalization of Lebesgue measure; a full definition is beyond our scope, but it is known that the Cantor set is a province under this measure (although the terminology is our own and this would be stated differently elsewhere). A thorough treatment of Hausdorff measures can be found in [6].

B Properties of the interior operator

Here we will develop some of the theory needed to establish the properties we use of the interior operator on a measure algebra.

We first note that the relation $\sqsubseteq$ is well-behaved under taking countable unions:

Lemma B.1 *Suppose that $\langle X, \mu\rangle$ is a measure space, $E \subseteq X$ and $\langle S_n \rangle_{n<\omega}$ is a sequence of subsets of S such that $[S_n]_\mu \sqsubseteq [E]_\mu$ for all $n < \omega$.*

Then,

$$\left[\bigcup_{n<\omega} S_n\right] \sqsubseteq [E]_\mu.$$

Proof. We have that

$$\begin{aligned}\mu\left(\left(\bigcup_{n<\omega} S_n\right)\setminus E\right) &\leq \mu\left(\bigcup_{n<\omega}(S_n\setminus E)\right)\\ &\leq \sum_{n<\omega}\mu(S_n\setminus E)\\ &= \sum_{n<\omega} 0\\ &= 0.\end{aligned}$$

But this implies that $\left[\bigcup_{n<\omega} S_n\right] \sqsubseteq [E]_\mu$, as desired. □

If $\langle X,\mu\rangle$ is a measure space and $\mathcal{O}\subseteq\mathbb{A}_\mu$, $\bigsqcup\mathcal{O}$ denotes the *supremum* of $\mathcal{O}$, that is, the least $u\in\mathbb{A}_\mu$ such that $o\sqsubseteq u$ for all $o\in\mathcal{O}$. As we show below, if μ is σ-finite, $\bigsqcup\mathcal{O}$ is always defined[7].

Lemma B.2 *If $\langle X,\mu\rangle$ is a measure space and $\langle S_n\rangle_{n<\omega}$ is a sequence of subsets of X, then $\bigsqcup\langle[S_n]_\mu\rangle_{n<\omega}$ is defined and equals $\left[\bigcup_{n<\omega} S_n\right]$.*

Proof. Clearly $\left[\bigcup_{n<\omega} S_n\right]$ is an upper bound for $\bigsqcup\langle[S_n]_\mu\rangle_{n<\omega}$; Lemma B.1 guarantees that it is the least upper bound, since any element of $\mathbb{A}_\mu$ which is greater than all $[S_n]_\mu$ is also greater than their union. □

All operations on the measure algebra are essentially countable in the following sense:

Lemma B.3 *If $\langle X,\mathcal{A},\mu\rangle$ is a σ-finite measure space and $\mathcal{O}\subseteq\mathbb{A}_\mu$, then $\bigsqcup\mathcal{O}$ (the supremum of $\mathcal{O}$) is defined and there is a sequence $\langle e_n\rangle_{n<\omega}$ of elements of $\mathcal{O}$ such that*

$$\bigsqcup\mathcal{O} = \bigsqcup_{n<\omega} e_n.$$

Proof. Suppose $\mathcal{O}=\langle o_\xi\rangle_{\xi<\gamma}$, where γ is a possibly uncountable cardinal.

By cardinal induction, we can assume that $O_\xi=\bigsqcup_{\zeta<\xi} o_\xi$ is defined for all $\xi<\gamma$.

Let I be the set of all $\xi<\gamma$ such that $\mu(O_\zeta)<\mu(O_\xi)$ for all $\zeta<\xi$. Write I as an increasing sequence $I=\langle\iota_\xi\rangle_{\xi<\lambda}$.

One can see that for all $\chi\neq\zeta<\lambda$,

$$(O_{\iota_{\xi+1}}-O_{\iota_\xi})\sqcap(O_{\iota_{\zeta+1}}-O_{\iota_\zeta})=[\varnothing]_\mu$$

and each $O_{\iota_{\xi+1}}-O_{\iota_\xi}$ is of positive measure; since μ is σ-finite, it follows that there can be only countably many of them, and therefore I must be countable.

We claim that $\bigsqcup_{\xi\in I} O_\xi=\bigsqcup\mathcal{O}$. Note that this is sufficient to establish our result; by induction hypothesis, for each $\xi<\gamma$ we can write $O_\xi=\bigsqcup_{n<\omega} e^\xi_n$ with $e^\xi_n\in\mathcal{O}$, so that $\bigsqcup_{\xi\in I} O_\xi=\bigsqcup_{\xi\in I}\bigsqcup_{n<\omega} e^\xi_n$. Since I is countable the latter is a supremum over a countable set, as desired.

[7] Indeed, $\mathbb{A}_\mu$ is a complete Boolean algebra, a fact which was proven by Tarski in [16]. The same paper indicates that Jaskowski proved the result in the special case of the Lebesgue measure algebra in 1931, but did not publish the proof at the time.

Clearly $\bigsqcup_{\xi \in I} O_\xi \sqsubseteq \bigsqcup \mathcal{O}$, so to show that $\bigsqcup_{\xi \in I} O_\xi = \bigsqcup \mathcal{O}$ it suffices to prove that $\bigsqcup_{\xi \in I} O_\xi$ is an upper bound for $\mathcal{O}$.

Pick $\zeta < \gamma$ and consider the least ordinal ϑ such that $\mu(O_\vartheta) = \mu(O_\zeta)$. By the way we defined I we have that $\vartheta \in I$; but then $O_\zeta \sqsubseteq O_\vartheta$, and hence $O_\zeta \sqsubseteq \bigsqcup_{\xi \in I} O_\xi$. Since $o_\zeta \sqsubseteq O_\zeta$ and ζ was arbitrary, the claim follows. □

We are now ready to prove Proposition 2.4.

Proof. [Proof of Proposition 2.4] Let $\langle X, \mathcal{T}, \mu \rangle$ be a topological measure space and o an element of its measure algebra.

By Lemma B.3 there are countably many open sets U_n with $[U_n]_\mu \sqsubseteq o$ such that $o^\Box = \bigsqcup_{n<\omega} [U_n]_\mu$; by Lemma B.2, $\bigsqcup_{n<\omega} [U_n]_\mu = \left[\bigcup_{n<\omega} U_n\right]_\mu$. Then,

(i) $o^\Box$ is open, since $\bigcup_{n<\omega} U_n$ is an open set;

(ii) follows from the definition of the interior operator;

(iii) follows immediately from the fact that $o^\Box$ is open.

□

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