Goldblatt-Thomason-style Theorems for Graded Modal Language

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Abstract

We prove two main Goldblatt-Thomason-style Theorems for graded modal language in Kripke semantics: full Goldblatt-Thomason Theorem for elementary classes and relative Goldblatt-Thomason Theorem within the class of finite transitive frames. Two different semantic views on GML allow us to prove these results: neighborhood semantics and graph semantics. By neighborhood semantic view, we can define a natural generalization of Jankov-Fine formula for GML and establish relative Goldblatt-Thomason Theorem. By extracting graph semantics from Fine's completeness proof of GML (1972), we introduce a new notion of graded ultrafilter images and establish full Goldblatt-Thomason Theorem. Therefore we revive Fine's old idea in the new context of Goldblatt-Thomason-style characterization.

Keywords: graded modal logic, Goldblatt-Thomason theorem, graded Jankov-Fine formula, graph semantics, graded ultrafilter images.

1 Introduction

Graded modal logic (GML) is one of extended modal logics. It was originally proposed by Kit Fine [10] to express modal analogues to counting quantifiers $\exists_k x P(x)$ in first-order logic explored by A. Tarski [19]. The modal analogue to $\exists_k x P(x)$ is written as $\Diamond_k p$; it is true at a state w in a Kripke model iff the number of accessible p-worlds is at least k. In GML, we add the family $\{ \Diamond_k : k \in \omega \}$ of modal operators to the basic modal logic. In particular, $\Diamond_k$ is non-normal: $\Diamond_k \bot \leftrightarrow \bot$ $(k > 1)$ and $(\Diamond_k p \vee \Diamond_k q) \rightarrow \Diamond_k (p \vee q)$ are valid but $\Diamond_k (p \vee q) \rightarrow (\Diamond_k p \vee \Diamond_k q)$ is not valid $(k > 1)$. Such modalities are used when it comes to counting successors. For example, GML was applied to epistemic logic [25] and description logic [1].

The model theory for GML has been explored since the 1970s. Some normal graded modal logics were shown to be strongly complete by Kit Fine [10], and later canonical models were constructed by M. Fattorosi-Barnaba and C. Cerrato [8,9], F. de Caro [6], and C. Cerrato [3]. M. de Rijke [7] defined the notion of graded bisimulation and proved Van Benthem-style Characterization Theorem. De Rijke also noted that graded modalities $\Diamond_{n+2}$ are not definable in basic modal logic since they are not invariant under ordinary notion bisimulations. Hence GML is a proper extension of basic modal logic with more expressive power at the level of Kripke models. Recently, Ten Cate et.al. [21] observed that GML can describe finite tree models up to isomorphism. If we turn to Kripke frames, we can define the frame property "there exist at least two successors", by $\Diamond_2\top$, which is undefinable in basic modal logic. So, we can also state that GML is more expressive than basic modal logic at the level of Kripke frames.

Goldblatt-Thomason Theorem [12] allows us to characterize the modal definability of elementary classes by four frame constructions: generated subframes, disjoint unions, bounded morphic images, and ultrafilter extensions. By this theorem, we can state that the semantic essence for frame-definability of basic modal language consists of these four frame constructions. Later, Van Benthem [24] gave a model-theoretical proof of Goldblatt-Thomason Theorem. Since [12,24], Goldblatt-Thomason-style Theorem has been investigated also for extended modal logics: difference logic [11], hybrid logic [20], etc. The first author and Sato [18] (see also [17]) provided a uniform Goldblatt-Thomason-style characterization of frame definability for *any* modal language extended with a set of *normal* modal operators, whose accessibility relations are defined by Boolean combinations of a (binary) accessibility relation R and the equality, that is, by quantifier-free formulas.

As for Goldblatt-Thomason-style characterization of definability in GML, de Rijke asked the following question.

> Obvious questions to be answered next include the following: Can g-bisimulations be used to prove a Goldblatt-Thomason style results about the classes of frames definable in $\mathcal{L}_{\mathrm{GML}}$? [7, p.282]

As far as the authors know, we still lack Goldblatt-Thomason theorem for GML in Kripke semantics. In this paper, we provide two Goldblatt-Thomason-style Theorems, thus answering de Rijke's question positively (though we will not use the notion of g-bisimulation). One is Goldblatt-Thomason theorem for elementary classes of frames (Theorem 6.3), and the other is relative Goldblatt-Thomason theorem within the class of finite transitive frames (Theorem 4.3).

Our results for GML correspond to the results from [2, Theorem 3.21] and [2, Theorem 3.19] for basic modal logic. This generalization, however, is not straightforward. One of the main reasons is the non-normal character of $\Diamond_k$ mentioned above. In order to deal with modalities of this kind and obtain our main results, we need to apply two different semantic approaches to GML: *neighborhood semantics* and *graph semantics*. The neighborhood semantic view leads us to the notion of *g-bounded morphism* and to a natural generalization of Jankov-Fine formulas [2, pp.144-5] for GML. By applying these formulas, we establish relative Goldblatt-Thomason Theorem for GML in terms of generated subframes, (finite) disjoint unions, and g-bounded morphisms.

A difficulty for establishing full Goldblatt-Thomason Theorem for GML is also in finding an appropriate notion of ultrafilter extension for GML. The method used in [17] does not seem to work here, because of the non-normal behavior of $\Diamond_k$. However, following the observation of [18] about the completeness proof and Goldblatt-Thomason-style characterization, we can extract an appropriate frame construction from Kit Fine's original completeness proof for GML [10]. By analyzing his proof carefully, we demonstrate that Fine's construction allows us to define a new graph semantics for GML. Moreover, we rewrite Fine's notion of *canonical mapping* [10, p.518] as the new frame construction *graded ultrafilter image* via a graph frame defined on the set of all ultrafilters on (W, R). In other words, a key semantic idea for our full Goldblatt-Thomason-style Theorem already appeared implicitly in the first study of GML by Fine. So we *revive* Fine's old idea in the *new* context of Goldblatt-Thomason-style characterization for GML.

2 Kripke Semantics for Graded Modal Language

Graded modal language (*GML*, for short) consists of (i) a countable set Prop of proposition letters, (ii) $\wedge$, $\neg$, $\bot$, and (iii) a set $\{\, \Diamond_k : k \in \omega \,\}$ of modal operators called *graded modalities*. The set of formulas in GML is defined as follows:

$$\varphi ::= p \,|\, \bot \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, \Diamond_k \varphi,$$

where $p \in \mathsf{Prop}$ and $k \in \omega$. The intuitive meaning of $\Diamond_k\varphi$ is 'the number of accessible φ-worlds is at least k'. In addition to the usual abbreviations like $\rightarrow$, $\vee$, etc., we define $\Diamond\varphi := \Diamond_1\varphi$ and $\Box\varphi := \neg\Diamond\neg\varphi$. We also define $\Box^n p$ inductively as: $\Box^0 p := p$ and $\Box^{n+1}p$ $:= \Box\Box^n p$. By the *basic modal language*, we mean the sublanguage $\{\, \wedge, \neg, \bot, \Diamond_1 \,\} \cup \mathsf{Prop}$ of GML.

GML is interpreted in Kripke structures. A *Kripke frame* $\mathfrak{F}$ (or, just a *frame*) is a pair (W, R) of a non-empty set W and a binary relation $R \subseteq W^2$. A *Kripke model* $\mathfrak{M}$ (or, just a *model*) consists of a frame $\mathfrak{F} = (W, R)$ and a valuation $V : \mathsf{Prop} \to \mathcal{P}(W)$. The domain of a Kripke frame $\mathfrak{F}$ (or a Kripke model $\mathfrak{M}$) is denoted by $|\mathfrak{F}|$ (or $|\mathfrak{M}|$, respectively).

Given any model $\mathfrak{M} = (W, R, V)$, any $w \in W$ and any formula φ of GML, we define the *satisfaction relation* $\Vdash$ as standard except the clause for graded modalities:

$$\mathfrak{M}, w \Vdash \Diamond_k\varphi \text{ iff } \#(R(w) \cap [\![\varphi]\!]) \geq k,$$

where $R(w) := \{\, v \in W : wRv \,\}$, $[\![\varphi]\!]_{\mathfrak{M}} := \{\, v \in W : \mathfrak{M}, v \Vdash \varphi \,\}$ (when the context is clear, we usually drop the subscript), and $\#X$ means the cardinality of X. When $k = 0$, it is easy to see that $\Diamond_0\varphi$ is true at any state w of any model $\mathfrak{M}$. Remark that the satisfaction for $\Diamond\varphi := \Diamond_1\varphi$ is equivalent to $R(w) \cap [\![\varphi]\!] \neq \emptyset$. Based on the satisfaction relation, we can define the notion of *frame validity*, *frame definability*, *satisfiability*, etc.

as usual, cf. [2].

Fact 2.1 (Fine [10]) *The following formulas are valid in all frames:*

(i) $\Box(p \to q) \to (\Box p \to \Box q)$,

(ii) $\Diamond_k p \to \Diamond_l p$ $(l < k)$,

(iii) $\Diamond_k p \leftrightarrow \bigvee_{i=0}^{k}(\Diamond_i(p \wedge q) \wedge \Diamond_{k-i}(p \wedge \neg q))$,

(iv) $\Box(p \to q) \to (\Diamond_k p \to \Diamond_k q)$.

Here is another form of the truth condition for $\Diamond_k \varphi$ at $w \in W$ in (W, R, V):

$$\#(R(w) \cap [\![\varphi]\!]) \geq k \text{ iff } \exists X \subseteq R(w).\,(\#X = k \text{ and } X \subseteq [\![\varphi]\!]).$$

This form allows us to define *neighborhood maps* $\tau_k : W \to \mathcal{PP}(W)$ $(k \in \omega)$ for any given Kripke frame (W, R) as:

$$\tau_k(w) := \{\, Y \subseteq W \;:\; \exists X \subseteq R(w).\,(\#X = k \text{ and } X \subseteq Y)\,\}\,.$$

τ_k is closed under unions, i.e., $Y_1, Y_2 \in \tau_k(w)$ implies $Y_1 \cup Y_2 \in \tau_k(w)$, for any $w \in W$ and $Y_1, Y_2 \subseteq W$. Thus τ_k is *monotonic*, i.e., closed under set-inclusion $\subseteq$. However, $\tau_k(w)$ does not satisfy the following property in general (if $k > 1$): $Y_1 \cup Y_2 \in \tau_k(w)$ implies $Y_1, Y_2 \in \tau_k(w)$, for any $w \in W$ and $Y_1, Y_2 \subseteq W$. For any valuation V, it is clear that $\#(R(w) \cap [\![\varphi]\!]) \geq k$ iff $[\![\varphi]\!] \in \tau_k(w)$. This observation enables us to use the notion of *bounded morphism* between neighborhood structures for GML in the next section.

3 Preservation under Frame Constructions

Definition 3.1 (g-bounded morphism) *Given any $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$, we say that $f : W \to W'$ is a g-*bounded morphism *if: for any $k \in \omega$ and any $Y \subseteq W'$,*

$$\#(R(w) \cap f^{-1}[Y]) \geq k \text{ iff } \#(R'(f(w)) \cap Y) \geq k.$$

If there is a surjective g-bounded morphism from $\mathfrak{F}$ and $\mathfrak{F}'$, then we say that $\mathfrak{F}'$ is a g-bounded morphic image *of $\mathfrak{F}$ (notation: $\mathfrak{F} \twoheadrightarrow_g \mathfrak{F}'$).*

In terms of derived neighborhood structures, this definition can be rewritten as: $f^{-1}[Y] \in \tau_k(w)$ iff $Y \in \tau_k(f(w))$ for any $Y \subseteq W'$, where τ_k and τ'_k are the neighborhood structures derived from $\mathfrak{F}$ and $\mathfrak{F}'$, respectively. If we restrict our attention to $k = 1$ in the definition of g-bounded morphism, then we can easily obtain the notion of *bounded morphism* [2, p.59]:

Definition 3.2 *Given any $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$, we say that $f : W \to W'$ is a* bounded morphism *if $f[R(w)] = R'(f(w))$ for any $w \in W$.*

Proposition 3.3 *Assume that $\mathfrak{F} \twoheadrightarrow_g \mathfrak{F}'$. If $\mathfrak{F} \Vdash \varphi$, then $\mathfrak{F}' \Vdash \varphi$, for any φ of GML.*

Proof. We show the contrapositive implication. Assume $\mathfrak{F}' \nVdash \varphi$. That is, $(\mathfrak{F}', V'), w' \nVdash \varphi$ for some $w' \in |\mathfrak{F}'|$ and some valuation V'. Define a valuation V on $\mathfrak{F}$ by: $V(p) :=$

$f^{-1}[V'(p)]$ for any $p \in \mathsf{Prop}$. Put $\mathfrak{M} := (\mathfrak{F}, V)$ and $\mathfrak{M}' := (\mathfrak{F}', V')$. Then we can establish that $[\![\varphi]\!]_{\mathfrak{M}} = f^{-1}[[\![\varphi]\!]_{\mathfrak{M}'}]$ by induction on φ. Since f is surjective, $f(w) = w'$ for some $w \in |\mathfrak{F}|$. Therefore, we obtain $w \notin [\![\varphi]\!]_{\mathfrak{M}}$ hence $\mathfrak{F} \not\Vdash \varphi$. □

When we try to check that a given mapping is a g-bounded morphism, the equivalent notion of *locally injective bounded morphism* is quite helpful, since it does not involve any quantification over $Y \subseteq W'$.

Definition 3.4 *Given any* $\mathfrak{F} = (W, R)$ *and* $\mathfrak{F}' = (W', R')$*, we say that* $f : W \to W'$ *is* locally injective *if* $f \restriction R(w)$ *is injective for any* $w \in W$*.*

Proposition 3.5 *Given any* $\mathfrak{F} = (W, R)$*,* $\mathfrak{F}' = (W', R')$*, and* $f : W \to W'$*,* f *is a* g*-bounded morphism* iff f *is a locally injective bounded morphism.*

Proof. We only establish the left-to-right direction, since the converse direction is obvious. Suppose that f is a g-bounded morphism. It is easy to show that f is a bounded morphism (it suffices to use the clause of $k = 1$ in the definition of g-bounded morphism). We show that $f \restriction R(w)$ is injective. Let us fix any w_1, $w_2 \in R(w)$ with $w_1 \neq w_2$. Our goal is to establish that $f(w_1) \neq f(w_2)$. Since $\{ w_1, w_2 \} = R(w) \cap \{ w_1, w_2 \} \subseteq R(w) \cap f^{-1}[f[\{ w_1, w_2 \}]]$, we obtain $\#(R(w) \cap f^{-1}[f[\{ w_1, w_2 \}]]) \geq 2$. From the clause of $k = 2$ in the definition of g-bounded morphism it follows that $\#(R'(f(w)) \cap f[\{ w_1, w_2 \}]) \geq 2$. Since $R'(f(w)) = f[R(w)]$ (here we use the fact that f is a bounded morphism), we can establish that $\#(f[R(w)] \cap f[\{ w_1, w_2 \}]) \geq 2$. Equivalently, $\#f[\{ w_1, w_2 \}] \geq 2$. This means that $f(w_1) \neq f(w_2)$, as required. □

Surprisingly, this proof tells us that the clauses of $k = 1$ and 2 are enough to define the notion of g-bounded morphism.

Given any cardinal κ, let us denote by $\mathrm{NS}(\kappa)$ ('NS' means the number of successors) the following property of Kripke frames: $\#R(w) \geq \kappa$ for any $w \in W$. The next proposition shows the expressive strength of GML over the basic modal language.

Proposition 3.6 *Let* $k \in \omega$ *and* $k \geq 2$*. Then,* $\mathrm{NS}(k)$ *is undefinable in the basic modal language. However,* $\mathrm{NS}(k)$ *is definable in GML by* $\Diamond_k \top$*.*

Proof. The second part is easy to check. So we only show the first part. Define $\mathfrak{F}_k = (W, R)$ by: $W = \{ 1, ..., k \}^{<\omega}$ and $\langle l_1, ..., l_m \rangle R \langle r_1, ..., r_n \rangle$ iff $n = m + 1$ and $l_i = r_i$ for any i with $1 \leq i \leq m$. So $\mathfrak{F}_k$ is a tree with k-branches and ω-height. Define $\mathfrak{G}$ as a one-point reflexive frame $(\{ * \}, \{ (*, *) \})$. Let f be the unique mapping from W to $\{ * \}$. It is easy to see that f is a surjective bounded morphism. Then by the validity-preservation under bounded morphic images [2, Theorem 3.14 (iii)], we can establish the undefinablity of $\mathrm{NS}(k)$. □

Proposition 3.7 $\mathrm{NS}(\omega)$ *is definable in GML by* $\{ \Diamond_k \top : k \in \omega \}$*.*

However, irreflexivity (wRw fails for any $w \in W$) is still undefinable in GML.

Proposition 3.8 *Irreflexivity is undefinable in GML.*

Proof. Let us use the frame $\mathfrak{F}_1 = (W, R)$ and $\mathfrak{G}$ from the proof of Proposition 3.6 (we fix $k = 1$ here). Note that $\mathfrak{F}_1$ is irreflexive, but $\mathfrak{G}$ is not irreflexive. Then we can show that the unique surjective $f : W \to \{ * \}$ is a g-bounded morphism from $\mathfrak{F}_1$ to $\mathfrak{G}$. By Proposition 3.3, we can establish the undefinability of irreflexivity in GML. □

Definition 3.9 (Generated subframes) *Given any frames $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$, we say that $\mathfrak{F}'$ is a* generated subframe *of $\mathfrak{F}$ if* (i) $W' \subseteq W$, (ii) $R' = R \cap (W')^2$, (iii) $R(w') \subseteq W'$ *for any $w' \in W'$. We say that $\mathfrak{F}'$ is a* point-generated subframe *of $\mathfrak{F}$ by a* root *w in $\mathfrak{F}$ (notation: $\mathfrak{F}_w$) if $\mathfrak{F}'$ is the smallest generated subframe of $\mathfrak{F}$ whose domain contains w.*

Proposition 3.10 *If $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$, then $\mathfrak{F} \Vdash \varphi$ implies $\mathfrak{F}' \Vdash \varphi$, for any φ of GML.*

Definition 3.11 (Disjoint unions) *Let $\{ \mathfrak{F}_i : i \in I \}$ be a pairwise disjoint family of frames, where $\mathfrak{F}_i = (W_i, R_i)$. We define the disjoint union $\biguplus_{i \in I} \mathfrak{F}_i = (W, R)$ of $\{ \mathfrak{F}_i : i \in I \}$ as: $W = \bigcup_{i \in I} W_i$ and $R = \bigcup_{i \in I} R_i$.*

Proposition 3.12 *For any pairwise disjoint family $\{ \mathfrak{F}_i : i \in I \}$ of frames and any φ of GML, if $\mathfrak{F}_i \Vdash \varphi$ ($i \in I$) then $\biguplus_{i \in I} \mathfrak{F}_i \Vdash \varphi$.*

Note that we may assume that, up to isomorphism, any family of frames is pairwise disjoint.

Proposition 3.13 *Any frame $\mathfrak{F}$ is a g-bounded morphic image of the disjoint union of some family of generated subframes of $\mathfrak{F}$.*

Proof. It suffices to note that $\biguplus_{w \in |\mathfrak{F}|} \mathfrak{F}_w \twoheadrightarrow_g \mathfrak{F}$. □

4 Graded Modal Classes of Finite Transitive Frames

First, we define the graded Jankov-Fine formulas as follows.

Definition 4.1 *Let $\mathfrak{F}_w = (W, R)$ be a finite transitive frame with the root w. Put $W = \{w_0, \ldots, w_n\}$ and $w = w_0$. Associate each $w_i \in W$ with a new proposition letter p_i. Define $p_X := \bigvee \{ p_i : w_i \in X \}$ for each finite $X \subseteq W$. Let $\Box^+\varphi := \varphi \wedge \Box\varphi$. The* graded Jankov-Fine formula *$\varphi_{\mathfrak{F},w}$ is defined as the conjunction of all the following formulas:*

(i) p_0

(ii) $\Box(p_0 \vee \cdots \vee p_n)$

(iii) $\bigwedge \{ \Box^+(p_i \to \neg p_j) : i \neq j \}$

(iv) $\bigwedge \{ \Box^+(p_i \to \Diamond_k p_X) : X \in \tau_k(w_i) \}$

(v) $\bigwedge \{ \Box^+(p_i \to \neg\Diamond_k p_X) : X \notin \tau_k(w_i) \}$

Clearly, $\varphi_{\mathfrak{F},w}$ is true at w of (W, R) under the natural valuation $V(p_i) = \{ w_i \}$ (remark that $V(p_X) = X$ for finite $X \subseteq W$).

Lemma 4.2 *Let $\mathfrak{F} = (W, R)$ be a finite transitive point-generated frame with the root w. Then for any transitive $\mathfrak{G}$, the following are equivalent:*

(A) *$\varphi_{\mathfrak{F},w}$ is satisfiable in $\mathfrak{G}$,* (**B**) *there exists* $v \in |\mathfrak{G}|$ *such that* $\mathfrak{F}$ *is a g-bounded morphic image of* $\mathfrak{G}_v$.

Proof. We can easily show the direction from (B) to (A) by Proposition 3.10 and 3.13, since $\varphi_{\mathfrak{F},w}$ is satisfiable in $\mathfrak{F}$ at w under the natural valuation V such that $V(p_i) = \{ w_i \}$. Conversely, let us assume (A). It follows that $(\mathfrak{G}_v, U), v \Vdash \varphi_{\mathfrak{F},w}$ for some $v \in |\mathfrak{G}|$ and some valuation U on $|\mathfrak{G}_v|$. Now put $\mathfrak{G}_v = (G, S)$. By the conjuncts (ii) and (iii) of $\varphi_{\mathfrak{F},w}$, we have $G = \bigcup_{0 \le 1 \le n} U(p_i)$ and $U(p_i) \cap U(p_j) = \emptyset$ for any i, j with $i \neq j$, respectively. By the rootedness of $\mathfrak{F}$ and (iv), we can also establish that $U(p_i) \neq \emptyset$ for any i with $0 \le i \le n$. This allows us to define a *surjective* mapping $f : G \to W$.

Now we show that f is a g-bounded morphism. Consider $k \in \omega$ and $X \subseteq W$ and $x \in G$ (note that X is finite). Let us put $f(x) := w_i$. We show the equivalence: $\#(S(x) \cap f^{-1}[X]) \ge k$ iff $\#(R(w_i) \cap X) \ge k$. First, we establish the left-to-right direction. We show the contrapositive implication. So, suppose $\#(R(w_i) \cap X) < k$ hence $X \notin \tau_k(w_i)$. Then, by the conjunct (v) of $\varphi_{\mathfrak{F},w}$ and our assumption: $(\mathfrak{G}_v, U), v \Vdash \varphi_{\mathfrak{F},w}$, we deduce that $(\mathfrak{G}_v, U), v \Vdash \Box^+(p_i \to \neg\Diamond_k p_X)$. Since $x \in |\mathfrak{G}_v|$ and $\mathfrak{G}_v$ is still transitive, from $x \in U(p_i)$ we deduce that $(\mathfrak{G}_v, U), x \nVdash \Diamond_k p_X$, hence $\#(S(x) \cap U(p_X)) < k$. It is easy to show that $f^{-1}[X] = U(p_X)$ (recall $p_X := \bigvee \{ p_i : w_i \in X \}$ and X is finite). Therefore, $\#(S(x) \cap f^{-1}[X]) < k$, as desired. Second, we can also establish the right-to-left direction similarly to the argument above. It suffices to use the conjunct (iv) instead of (v). □

Theorem 4.3 *Let* C *be the class of all finite transitive frames and* $\mathsf{F} \subseteq \mathsf{C}$. *Then* F *is definable by a set of formulas in GML within* C iff F *is closed under taking* (i) *generated subframes,* (ii) (*finite*) *disjoint unions,* (iii) *g-bounded morphic images.*

Proof. We can easily establish the left-to-right direction by Propositions 3.3, 3.10 and 3.12. Conversely, suppose that F satisfies the closure conditions. Define $\mathrm{Log}(\mathsf{F}) := \{ \varphi : \mathsf{F} \Vdash \varphi \}$. Let us show that $\mathrm{Log}(\mathsf{F})$ defines F within C. Consider $\mathfrak{F} \in \mathsf{C}$, i.e., $\mathfrak{F}$ is finite and transitive. We show the equivalence: $\mathfrak{F} \in \mathsf{F}$ iff $\mathfrak{F} \Vdash \mathrm{Log}(\mathsf{F})$. The left-to-right direction is easy to show. Let us establish the right-to-left direction. Assume $\mathfrak{F} \Vdash \mathrm{Log}(\mathsf{F})$. We subdivide our argument into the following two cases: (a) $\mathfrak{F}$ is point-generated and (b) $\mathfrak{F}$ is not point-generated.

First, let us consider the case (a). Let w be the root of $\mathfrak{F}$. Consider the Jankov-Fine formula $\varphi_{\mathfrak{F},w}$ of $\mathfrak{F}$ and w (note that $\mathfrak{F}$ is finite, transitive, and point-generated). Since $\varphi_{\mathfrak{F},w}$ is satisfiable in $\mathfrak{F}$ (i.e., $\mathfrak{F} \nVdash \neg\varphi_{\mathfrak{F},w}$), we have $\neg\varphi_{\mathfrak{F},w} \notin \mathrm{Log}(\mathsf{F})$. Then there exists $\mathfrak{G} \in \mathsf{F}$ such that $\varphi_{\mathfrak{F},w}$ is satisfiable in $\mathfrak{G}$. From Lemma 4.2 it follows that $\mathfrak{F}$ is a g-bounded morphic image of $\mathfrak{G}_v$ for some v in $\mathfrak{G}$. Since $\mathfrak{G} \in \mathsf{F}$, $\mathfrak{G}_v \in \mathsf{F}$ by the closure property (i) of F. Therefore, from the closure property (iii) of F and $\mathfrak{G}_v \twoheadrightarrow_g \mathfrak{F}$ we deduce that $\mathfrak{F} \in \mathsf{F}$, as required.

Second, let us consider the case (b). By Proposition 3.13 and our assumption, it is enough to show that each point generated subframe of $\mathfrak{F}$ is in F. But each of these frames validates $\mathrm{Log}(\mathsf{F})$ by Proposition 3.10, and hence belongs to F by the same argument as in the case (a). □

Let us say that $\mathfrak{F} = (W, R)$ is *m-transitive* if $R^{\le m} := \bigcup_{1 \le k \le m} R^k$ is transitive, where R^n is defined inductively by: $R^1 = R$ and $R^{n+1} := R^n \circ R$. We can define m-transitivity by $\Box^{\leq m}p \to \Box^{\leq m}\Box^{\leq m}p$, where $\Box^{\leq m}p := \Box p \wedge \cdots \wedge \Box^m p$. We can easily generalize Theorem 4.3 to cover the class C of all finite *m-transitive* frames, since it suffices to modify our Jankov-Fine formula as follows: replace each occurrence of $\Box$ with $\Box^{\leq m}$.

Van Benthem [23, p.29] showed that the assumption 'C is transitive' is crucial in the corresponding theorem [2, Theorem 3.21] for basic modal logic. We can use the same example to show that the transitivity assumption for C is also crucial in GML. Intuitively, this is because his proof uses frames, where every world has at most one successor [23, p.29], and so $\Diamond_k$ $(k > 1)$ does not matter. Let F be the class of finite linear orders with immediate succession as in [23, p.29] and F' be the finite disjoint union closure of F. It is easy to see that F' is closed under taking finite disjoint unions, generated subframes and also *g-bounded morphic images*. By the same argument, however, we can show that a one-point reflexive frame validates $\mathrm{Log}(\mathsf{F}') = \{\, \varphi : \mathsf{F}' \Vdash \varphi \,\}$. This shows the indispensability of transitivity in Theorem 4.3.

5 Graph Semantics and Graded Ultrafilter Images

5.1 Graph Semantics and Fine Mapping

Definition 5.1 *Define $X \subseteq_\omega Y$ if $X \subseteq Y$ and $\#X < \omega$.*

Definition 5.2 (Graph semantics for GML) *A* (directed) graph frame *is a pair of a non-empty set W and a family $(R_k)_{k\in\omega}$ of binary relations on W such that if $k > l$ then $R_k \subseteq R_l$. A graph model is a pair of a graph frame and a valuation on it. Given any graph model $(W, (R_k)_{k\in\omega}, V)$, we define the satisfaction relation $w \models_V \varphi$ as follows:*

$$\begin{array}{lll} w \models_V p & \text{iff} & w \in V(p), \\ w \models_V \bot & & \text{Never}, \\ w \models_V \neg\varphi & \text{iff} & w \not\models_V \varphi, \\ w \models_V \varphi \wedge \psi & \text{iff} & w \models_V \varphi, \text{ and } w \models_V \psi \\ w \models_V \Diamond_k\varphi & \text{iff} & \exists\, X \subseteq_\omega W.\ \exists\, l : X \to \omega. \\ & & \left(\sum_{v\in X} l(v) \geq k \,\text{and}\, \forall\, v \in X.\, (wR_{l(v)}v \,\text{and}\, v \in |\varphi|)\right), \end{array}$$

where $|\varphi| := \{\, w \in W : w \models_V \varphi \,\}$ (if $X = \emptyset$, we put $\sum_{v\in X} l(v) = 0$). If $w \models_V \varphi$ for all $w \in W$ and all $V : \mathsf{Prop} \to \mathcal{P}(W)$, we say that φ is valid *on $(W, (R_k)_{k\in\omega})$ and denote it by $(W, (R_k)_{k\in\omega}) \models \varphi$.*

For any $k \in \omega$, we can easily check that $w \models_V \Diamond_k\varphi$ is also equivalent to:

$$\exists\, X \subseteq_\omega W.\ \exists\, l : X \to \omega \setminus \{\, 0 \,\}.\ \left(\sum_{v\in X} l(v) \geq k \,\text{and}\, \forall\, v \in X.\, (wR_{l(v)}v \,\text{and}\, v \in |\varphi|)\right).$$

In this sense, R_0 does not play any role in the truth condition of $\Diamond_0\varphi$. As in Kripke semantics for GML, $(W, (R_k)_{k\in\omega}) \models \Diamond_0\varphi$ for any graph frame $(W, (R_k)_{k\in\omega})$, since it suffices to take $\emptyset \subseteq_\omega W$ and the empty function from $\emptyset$ to ω as our witness.

The following notion is our renewal of Fine's *canonical mapping* [10, pp.518-9].

Definition 5.3 (Fine mapping) *Let $(W, (R_k)_{k\in\omega})$ be a graph frame and $\mathfrak{G} = (G, S)$ a Kripke frame. We say that $f : G \to W$ is a* Fine mapping *if, for any $n \in \omega$, any $x \in G$ and any $w \in W$,*

$$\#\{\, y \in G \,:\, f(y) = w \text{ and } xSy \,\} \geq n \;\text{ iff }\; f(x)R_n w.$$

We call the left-to-right direction (**Forth**) *and the right-to-left direction* (**Back**)*.*

A surjective Fine mapping allows us to associate a graph frame with a Kripke frame in a validity-preserving way (Proposition 5.7). Before showing this, let us see some examples of Fine mapping.

Example 5.4 (Fine [10]) Let $(W, (R_k)_{k\in\omega})$ be a graph frame. Let us define a Kripke frame $\mathfrak{G} = (G, S_1)$ by (i) $G := W \times \omega$ and (ii) $(w, l)S_1(w', k)$ iff $k > l$ and $wR_{k-l}w'$. Then the projection $\pi_1 : W \times \omega \to W$ is a Fine mapping. This is verified as follows: Consider any $(w, l) \in G$ and $w' \in W$. Take any $n \in \omega$. Then:

$$\begin{aligned} &\#\{\, (w', k) \in G \,:\, \pi_1((w', k)) = w' \text{ and } (w, l)S_1(w', k) \,\} \geq n \\ \text{iff } &\#\{\, (w', k) \in G \,:\, k > l \text{ and } wR_{k-l}w' \,\} \geq n \text{ iff } wR_n w' \end{aligned}$$

Let us check the latter equivalence. First, let us show the left-to-right direction. Assume that $\#\{\, (w', k) \in G \,:\, k > l \text{ and } wR_{k-l}w' \,\} \geq n$. Fix $k_1, \ldots, k_n$ such that $(w', k_i) \in G$ and $k_i > l$ and $wR_{k_i-l}w'$ $(1 \leq i \leq n)$. Since $\max\{\, k_i - l : 1 \leq i \leq n \,\} \geq n$, we have $wR_n w'$ by $R_{k_1} \subseteq R_{k_2}$ $(k_1 > k_2)$. In order to establish the right-to-left direction, assume $wR_n w'$. From $R_{k_1} \subseteq R_{k_2}$ $(k_1 > k_2)$ we deduce that $wR_{n-1}w'$, $\cdots$, $wR_1 w'$. Then it is easy to see that $(w', l+1)$, $\ldots$, $(w', l+n)$ belong to $\{\, (w', k) \in G \,:\, k > l \text{ and } wR_{k-l}w' \,\}$.

Example 5.5 (Fine [10]) In Example 5.4, we can replace S_1 with the following S_2: $(w, l)S_2(w', k)$ iff $wR_k w'$. Then the projection $\pi_1 : W \times \omega \to W$ is still a Fine mapping. This is verified as follows: Consider any $(w, l) \in G$ and $w' \in W$. Take any $n \in \omega$. Then:

$$\begin{aligned} &\#\{\, (w', k) \in G \,:\, \pi_1((w', k)) = w' \text{ and } (w, l)S_2(w', k) \,\} \geq n \\ \text{iff } &\#\{\, (w', k) \in G \,:\, wR_k w' \,\} \geq n \text{ iff } wR_n w' \quad (\text{by } R_{k_1} \subseteq R_{k_2}\ (k_1 > k_2)) \end{aligned}$$

We will see some specific application of these constructions later on, in Examples 5.22 and 5.24.

Lemma 5.6 *Let $(W, (R_k)_{k\in\omega})$ be a graph frame and $\mathfrak{G} = (G, S)$ a Kripke frame, and V a valuation on W. Assume that $f : G \to W$ is a Fine mapping. Define a valuation V' on $\mathfrak{G}$ by $V'(p) = f^{-1}[V(p)]$. Then for any formula φ and $x \in G$,*

$$(\mathfrak{G}, V'), x \Vdash \varphi \;\text{ iff }\; f(x) \models_V \varphi.$$

Proof. By induction on φ. It suffices to check the case where $\varphi \equiv \Diamond_k\psi$. Consider any $x \in G$ and put $w := f(x)$. First, assume that $w \models_V \Diamond_k\psi$. That is,

$$\exists\, X \subseteq_\omega W.\, \exists\, l : X \to \omega.\, \left(\sum\nolimits_{v\in X} l(v) \geq k \,\text{and}\, \forall\, v \in X.\, (wR_{l(v)}v \,\text{and}\, v \in |\psi|)\right).$$

By definition of f and I.H. ($[\![\psi]\!]_{(\mathfrak{G},V')} = f^{-1}[|\psi|]$, the inverse image of $|\psi|$ by f), it follows that:

$$\exists\, X \subseteq_\omega W.\, \exists\, l : X \to \omega. \tag{†}$$
$$\left(\sum_{v\in X} l(v) \geq k \,\text{and}\, \forall\, v \in X.\, \left(\#(S(x) \cap f^{-1}[\{\, v\,\}]) \geq l(v) \text{ and } f^{-1}[\{\, v\,\}] \subseteq [\![\psi]\!]_{(\mathfrak{G},V')}\right)\right).$$

Then it follows that $\#(S(x) \cap [\![\psi]\!]_{(\mathfrak{G},V')}) \geq k$ hence $(\mathfrak{G}, V'), x \Vdash \Diamond_k\psi$, as required.

Conversely, assume that $(\mathfrak{G}, V'), x \Vdash \Diamond_k\psi$, i.e., $\#(S(x) \cap [\![\psi]\!]_{(\mathfrak{G},V')}) \geq k$ (we drop the subscript $(\mathfrak{G}, V')$ from $[\![\psi]\!]_{(\mathfrak{G},V')}$ below and write $[\![\psi]\!]'$). It suffices to derive (†). Note that $S(x) \cap [\![\psi]\!]'$ consists of the partitions $\{\, f^{-1}[\{\, v\,\}] : v \in f[S(x) \cap [\![\psi]\!]']\,\}$. The number of these partitions is $\alpha := \#f[S(x) \cap [\![\psi]\!]']$. If $\alpha \geq k$, we can easily derive (†): it suffices to choose some $X \subseteq_\omega f[S(x) \cap [\![\psi]\!]']$ such that $\#X = k$ and define $l : X \to \omega$ by $l(v) := \#(S(x) \cap f^{-1}[\{\, v\,\}])$. So, assume that $\alpha < k$. Then $\#(S(x) \cap [\![\psi]\!]') \geq k$ allows us to derive (†): let us put $X := f[S(x) \cap [\![\psi]\!]']$ and define $l(v) := \#(S(x) \cap f^{-1}[\{\, v\,\}])$. □

Proposition 5.7 *Let $(W, (R_k)_{k\in\omega})$ be a graph frame and $\mathfrak{G} = (G, S)$ a frame. If $f : G \to W$ is a surjective Fine mapping and $\mathfrak{G} \Vdash \varphi$, then $(W, (R_k)_{k\in\omega}) \models \varphi$.*

Proof. Assume $w \not\models_V \varphi$ for some V on W and some $w \in W$. Since f is surjective, $f(x) = w$ for some $x \in G$. Define V' on $\mathfrak{G}$ by $V'(p) := f^{-1}[V(p)]$ for all $p \in \mathsf{Prop}$. By Proposition 5.6, $(\mathfrak{G}, V'), x \nVdash \varphi$ hence $\mathfrak{G} \nVdash \varphi$. □

This proposition shows soundness of graph semantics for GML.

Corollary 5.8 *All the formulas from Fact 2.1 are valid in any graph frame $(W, (R_k)_{k\in\omega})$.*

Proof. By Proposition 5.7 and Example 5.4. □

5.2 Ultrafilter Graph Model and Graded Ultrafilter Images

Given any $\mathfrak{F} = (W, R)$ and any $k \in \omega$, define $m_k : \mathcal{P}(W) \to \mathcal{P}(W)$ by $m_k(X) := \{\, w \in W : \#(R(w) \cap X) \geq k\,\}$. Let us write $m_R(X) := m_1(X)$ and define $l_R : \mathcal{P}(W) \to \mathcal{P}(W)$ by $l_R(X) := W \setminus m_R(W \setminus X)$.

Definition 5.9 *Let $X, Y \subseteq W$. We define $X \Rightarrow Y := (W \setminus X) \cup Y$.*

Proposition 5.10 *For any $X \subseteq W$,*

(i) $l_R(X \Rightarrow Y) \cap l_R(X) \subseteq l_R(Y)$.

(ii) $m_k(X) \subseteq m_l(X)$ $(l < k)$.

(iii) $m_k(X) = \bigcup_{i=0}^{k}(m_i(X \cap Y) \cap m_{k-i}(X \cap (W \setminus Y)))$ *for any $Y \subseteq W$.*

(iv) $l_R(X \Rightarrow Y) \cap m_k(X) \subseteq m_k(Y)$.

Proof. It is easy to see that $\mathfrak{M}, w \Vdash \Diamond_k \varphi$ iff $w \in m_k([\![\varphi]\!])$, i.e., $[\![\Diamond_k \varphi]\!] = m_k([\![\varphi]\!])$. Then all four items are clear from Fact 2.1. □

Definition 5.11 *We define the binary relation* $R_k^{\mathfrak{ue}}$ *on the set* $\mathrm{Uf}(W)$ *of all ultrafilters on* W *as:* $\mathcal{U} R_k^{\mathfrak{ue}} \mathcal{U}'$ iff $X \in \mathcal{U}'$ *implies* $m_k(X) \in \mathcal{U}$, *for any* $X \subseteq W$.

Proposition 5.12 $(\mathrm{Uf}(W), (R_k^{\mathfrak{ue}})_{k \in \omega})$ *is a graph frame.*

Proof. It suffices to check that $R_k \subseteq R_l$ $(l < k)$. This follows trivially from Proposition 5.10 (ii). □

Then we can define the final frame construction to characterize the definability of GML for elementary classes as follows.

Definition 5.13 (Graded ultrafilter images) *Let* $\mathfrak{F} = (W, R)$ *and* $\mathfrak{G} = (G, S)$ *be frames. We say that* $f : G \to \mathrm{Uf}(W)$ *is a* graded ultrafilter mapping *if* f *is a Fine mapping from* $\mathfrak{G}$ *to* $(\mathrm{Uf}(W), (R_k^{\mathfrak{ue}})_{k \in \omega})$. $\mathfrak{F}$ *is a* graded ultrafilter image *of* $\mathfrak{G}$ *if there exists a graded ultrafilter mapping* $f : G \to \mathrm{Uf}(W)$ *such that* f *is surjective.*

The rest of this subsection is devoted to establishing the preservation result for graded ultrafilter images. Our strategy is as follows. First, we show the implication: if $(\mathrm{Uf}(W), (R_k^{\mathfrak{ue}})_{k \in \omega}) \models \varphi$ then $(W, R) \Vdash \varphi$ (Proposition 5.20). Only for this purpose, we use the notion called *truth-set function* originating from Fine [10] [1]. Second, by combining Proposition 5.7 with the implication above, we will establish our desired preservation result (Theorem 5.21).

Definition 5.14 (Fine [10]) *Given any frame* (W, R) *and an ultrafilter* $\mathcal{U}$ *on* W, *we define the* truth-set function $\mathrm{T}_{\mathcal{U}}(-) : \mathcal{P}(W) \to \mathcal{P}(\mathrm{Uf}(W) \times \omega)$ *by:*

$$\mathrm{T}_{\mathcal{U}}(X) := \{\, (\mathcal{U}', l) \,:\, l > 0 \text{ and } X \in \mathcal{U}' \text{ and } \mathcal{U} R_l^{\mathfrak{ue}} \mathcal{U}' \,\}.$$

Proposition 5.15 *For any* $X, Y \subseteq W$,

(i) $\mathrm{T}_{\mathcal{U}}(X \cap Y) \cap \mathrm{T}_{\mathcal{U}}(X \cap (W \setminus Y)) = \emptyset$.

(ii) $\mathrm{T}_{\mathcal{U}}(X) = \mathrm{T}_{\mathcal{U}}(X \cap Y) \cup \mathrm{T}_{\mathcal{U}}(X \cap (W \setminus Y))$.

Lemma 5.16 *For any* $(k, \mathcal{U}) \in \omega \times \mathrm{Uf}(W)$, $m_k(X) \in \mathcal{U}$ *implies* $\#\mathrm{T}_{\mathcal{U}}(X) \geq k$.

Proof. See Appendix A. □

Lemma 5.17 *For any* $(k, \mathcal{U}) \in \omega \times \mathrm{Uf}(W)$, $\#\mathrm{T}_{\mathcal{U}}(X) \geq k$ *implies* $m_k(X) \in \mathcal{U}$.

Proof. See Appendix A. □

Lemma 5.18 *For any* $k \in \omega$ *and any* $\mathcal{U} \in \mathrm{Uf}(W)$, $\#\mathrm{T}_{\mathcal{U}}(X) \geq k$ iff *:*

$$\exists\, \mathbb{X} \subseteq_{\omega} \mathrm{Uf}(W).\ \exists\, l : \mathbb{X} \to \omega.\ \Big(\textstyle\sum_{\mathcal{V} \in \mathbb{X}} l(\mathcal{V}) \geq k \text{ and } \forall\, \mathcal{V} \in \mathbb{X}.\, (\mathcal{V}, l(\mathcal{V})) \in \mathrm{T}_{\mathcal{U}}(X) \Big).$$

[1] If the reader checks Fine's paper [10], he might first feel that the truth-set function $\mathrm{T}_{\mathcal{U}}(-)$ plays the main role in Fine's completeness proof. However, from our viewpoint the graph semantics is the most essential in his proof.

Proof. The right-to-left direction is easy to show by Proposition 5.10 (ii). So let us establish the left-to-right direction. Assume $\#\mathrm{T}_{\mathcal{U}}(X) \geq k$. Remark that $\mathrm{T}_{\mathcal{U}}(X)$ consists of the partitions $\{\, \pi_1^{-1}[\{\, \mathcal{V}\,\}] \cap \mathrm{T}_{\mathcal{U}}(X) : \mathcal{V} \in \pi_1[\mathrm{T}_{\mathcal{U}}(X)]\,\}$, where $\pi_1 : \mathrm{Uf}(W) \times \omega \to \mathrm{Uf}(W)$ is the projection [2]. We can regard $\#\pi_1[\mathrm{T}_{\mathcal{U}}(X)]$ as the number of all these partitions. If $\#\pi_1[\mathrm{T}_{\mathcal{U}}(X)] \geq k$, we are done: it suffices to choose some $\mathbb{X} \subseteq_\omega \pi_1[\mathrm{T}_{\mathcal{U}}(X)]$ such that $\#\mathbb{X} = k$ and define $l : \mathbb{X} \to \omega$ such that $(\mathcal{V}, l(\mathcal{V})) \in \pi_1^{-1}[\{\, \mathcal{V}\,\}] \cap \mathrm{T}_{\mathcal{U}}(X)$. So let us consider the case where $\#\pi_1[\mathrm{T}_{\mathcal{U}}(X)] < k$. Define $\mathbb{X} := \pi_1[\mathrm{T}_{\mathcal{U}}(X)]$. We divide our argument into the following two cases: (a) there exists $\mathcal{V} \in \mathbb{X}$ such that $\#\pi_1^{-1}[\{\, \mathcal{V}\,\}] \cap \mathrm{T}_{\mathcal{U}}(X) \geq \omega$, (b) for all $\mathcal{V} \in \mathbb{X}$, $\#\pi_1^{-1}[\{\, \mathcal{V}\,\}] \cap \mathrm{T}_{\mathcal{U}}(X) < \omega$. First, let us consider the case (a). Fix such $\mathcal{V} \in \mathbb{X}$. Then for our purpose, it suffices to choose some $l : \mathbb{X} \to \omega$ such that $l(\mathcal{V}) \geq k$. Finally, let us turn to the case (b). For any $\mathcal{V} \in \mathbb{X}$, we can define $l(\mathcal{V}) := \max\{\, n : (\mathcal{V}, n) \in \mathrm{T}_{\mathcal{U}}(X)\,\}$. Then from $\#\mathrm{T}_{\mathcal{U}}(X) \geq k$ it is clear that $\sum_{\mathcal{V}\in\mathbb{X}} l(\mathcal{V}) \geq k$. □

Proposition 5.19 *Given any Kripke model $\mathfrak{M} := (W, R, V)$, define the graph model $(\mathrm{Uf}(W), (R_k^{\mathfrak{ue}})_{k\in\omega}, V^{\mathfrak{ue}})$, where $V^{\mathfrak{ue}}(p) := \{\mathcal{U} \in \mathrm{Uf}(W) : V(p) \in \mathcal{U}\}$. Then for any φ and any $\mathcal{U} \in \mathrm{Uf}(W)$,*

$$[\![\varphi]\!] \in \mathcal{U} \text{ iff } \mathcal{U} \models_{V^{\mathfrak{ue}}} \varphi.$$

Proof. By induction on φ. It suffices to consider the case where $\varphi \equiv \Diamond_k\psi$. Then $[\![\Diamond_k\psi]\!] \in \mathcal{U}$ iff $m_k([\![\psi]\!]) \in \mathcal{U}$ iff $\#\mathrm{T}_{\mathcal{U}}([\![\psi]\!]) \geq k$ by Lemma 5.16 and Lemma 5.17. Equivalently, by Lemma 5.18 and the definition of $\mathrm{T}_{\mathcal{U}}$, we obtain:

$$\exists\, \mathbb{X} \subseteq_\omega \mathrm{Uf}(W).\ \exists\, l : \mathbb{X} \to \omega.\ \left(\sum_{\mathcal{V}\in\mathbb{X}} l(\mathcal{V}) \geq k \text{ and } \forall\, \mathcal{V} \in \mathbb{X}.\ (\mathcal{U} R^{\mathfrak{ue}}_{l(\mathcal{V})} \mathcal{V} \text{ and } [\![\psi]\!] \in \mathcal{V})\right).$$

By I.H. ($[\![\psi]\!] \in \mathcal{V}$ iff $\mathcal{V} \in |\psi|$), we establish $\mathcal{U} \models_{V^{\mathfrak{ue}}} \Diamond_k\varphi$, as required. □

Proposition 5.20 *For any $\mathfrak{F} = (W, R)$, if $(\mathrm{Uf}(W), (R_k^{\mathfrak{ue}})_{k\in\omega}) \models \varphi$, then $\mathfrak{F} \Vdash \varphi$.*

Proof. Assume $\mathfrak{F} \nVdash \varphi$, i.e., $(\mathfrak{F}, V), w \nVdash \varphi$ for some V and some $w \in W$. Take the principal ultrafilter $\mathcal{U}_w := \{\, D \subseteq W : w \in D\,\}$. It follows that $[\![\varphi]\!] \notin \mathcal{U}_w$. It follows from Proposition 5.19 that $\mathcal{U}_w \not\models_{V^{\mathfrak{ue}}} \varphi$; hence $(\mathrm{Uf}(W), (R_k^{\mathfrak{ue}})_{k\in\omega}) \not\models \varphi$. □

Theorem 5.21 *If $\mathfrak{F}$ is a graded ultrafilter image of $\mathfrak{G}$, then $\mathfrak{G} \Vdash \varphi$ implies $\mathfrak{F} \Vdash \varphi$.*

Proof. By Proposition 5.7 and Proposition 5.20. □

Example 5.22 By definition of S_1 in Example 5.4, $\mathfrak{G}$ is irreflexive. Let us consider an example of graded ultrafilter images. Take a one-point reflexive frame $\mathfrak{F} = (\{\, *\,\}, \{\, (*, *)\,\})$. Then $\mathrm{Uf}(|\mathfrak{F}|)$ consists only of the principal ultrafilter $\mathcal{U}_*$ generated by $*$. We have $\mathcal{U}_* R_0^{\mathfrak{ue}} \mathcal{U}_*$ and $\mathcal{U}_* R_1^{\mathfrak{ue}} \mathcal{U}_*$ by definition. However, if $k \geq 2$, $\mathcal{U}_* R_k^{\mathfrak{ue}} \mathcal{U}_*$ fails. By the construction of Example 5.4, we can construct Kripke frame $\mathfrak{G} := (\{\, \mathcal{U}_*\,\} \times \omega, S_1)$.

[2] Remark that $\mathrm{T}_{\mathcal{U}}(X)$ is contained in (the union of) $\left\{\, \pi_1^{-1}[\{\, \mathcal{V}\,\}] : \mathcal{V} \in \pi_1[\mathrm{T}_{\mathcal{U}}(X)]\,\right\}$. However, $\left\{\, \pi_1^{-1}[\{\, \mathcal{V}\,\}] \cap \mathrm{T}_{\mathcal{U}}(X) : \mathcal{V} \in \pi_1[\mathrm{T}_{\mathcal{U}}(X)]\,\right\}$ gives us a partition of $\mathrm{T}_{\mathcal{U}}(X)$.

It is easy to see that $\mathfrak{G}$ is isomorphic to (ω, Suc^{-1}), where Suc^{-1} is the inverse of the successor relation Suc on ω.

This example also shows the undefinability of irreflexivity in GML by Theorem 5.21. Also by Example 5.22 and Theorem 5.21, we can establish:

Proposition 5.23 *The existence of a distinct predecessor (for any w, there exists w' such that $w'Rw$ and $w \neq w'$) is undefinable in GML.*

Example 5.24 In Example 5.22, let us start from the one point *irreflexive* frame $\mathfrak{F}' := (\{*\}, \emptyset)$. While $\mathrm{Uf}(|\mathfrak{F}|)$ consists only of the principal ultrafilter $\mathcal{U}_*$ generated by $*$ as before, $\mathcal{U}_* R_0^{\mathfrak{ue}} \mathcal{U}_*$ holds but $\mathcal{U}_* R_k^{\mathfrak{ue}} \mathcal{U}_*$ fails $(k > 0)$. Here let us use the definition S_2 of Example 5.5. Then $\mathfrak{G} := (\{\mathcal{U}_*\} \times \omega, S_2)$ is isomorphic to (ω, R'), where $nR'm$ iff $m = 0$. Therefore a reflexive state is accessible from all the states in $\mathfrak{G}$. However, this is not the case in $\mathfrak{F}'$. This example uses the relation $R_0^{\mathfrak{ue}}$ crucially.

By Example 5.24 and Theorem 5.21, we can establish:

Proposition 5.25 $\forall w.\, \exists w'.\, (wRw' \,\mathrm{and}\, w'Rw')$ *is undefinable in GML.*

6 Goldblatt-Thomason-style Characterization of Elementary Graded Modal Classes

In this section we use some notions from first-order model theory, e.g., elementary embedding, ω-saturation, etc. The reader unfamiliar with them can refer to [4]. The original Goldblatt-Thomason Theorem for basic modal logic was proved via duality between algebras and frames [12]. The proof of our Goldblatt-Thomason Theorem for GML modifies the model-theoretic proof given by Van Benthem [24] for basic modal logic.

Definition 6.1 *Let $\mathfrak{F} = (W, R)$ be a generated subframe with a root w. We expand our language GML with the (possibly uncountable) set $\{\, p_X : X \subseteq W \,\}$ of new proposition letters. Let Δ be the set consisting of:*

$$\begin{aligned} p_{X\cap Y} &\leftrightarrow p_X \wedge p_Y, \\ p_{W\setminus X} &\leftrightarrow \neg p_X, \\ p_{m_k(X)} &\leftrightarrow \Diamond_k p_X, \\ p_W&, \end{aligned}$$

where $X, Y \subseteq W$ and $k \in \omega$. Then we define $\Delta_{\mathfrak{F}}$ as follows:

$$\Delta_{\mathfrak{F}} := \{\, p_{\{w\}} \,\} \cup \{\, \Box^n \varphi : \varphi \in \Delta \text{ and } n \in \omega \,\}.$$

Note that $\Delta_{\mathfrak{F}}$ is satisfiable in $\mathfrak{F}$ under the natural valuation V such that $V(p_X) = X$. Let F be an elementary class of frames. Similarly to our graded Jankov-Fine formula in Definition 4.2, by this 'complete description' of $\mathfrak{F}$, for a given $\mathfrak{G} \in \mathsf{F}$ such that $\Delta_{\mathfrak{F}}$ is satisfiable in $\mathfrak{G}$, we can extract the following semantic information.

Lemma 6.2 *Let* F *be an elementary class of frames and* $\mathfrak{F} = (W, R)$ *a generated subframe with a root* w. *Also let* $\mathfrak{G} \in \mathsf{F}$. *If* $\Delta_{\mathfrak{F}}$ *is satisfiable in* $\mathfrak{G}$, *then there exists some* $v \in |\mathfrak{G}|$ *and some elementary extension* $(\mathfrak{G}_v)^*$ *of* $\mathfrak{G}_v$ *such that* $\mathfrak{F}$ *is a graded ultrafilter image of* $(\mathfrak{G}_v)^*$.

Proof. Let us assume that $\Delta_{\mathfrak{F}}$ is satisfiable in $\mathfrak{G}$. Thus we have $(\mathfrak{G}, V), v \Vdash \Delta_{\mathfrak{F}}$ for some valuation V and $v \in |\mathfrak{G}|$. It follows that $(\mathfrak{G}_v, V_1), v \Vdash \Delta_{\mathfrak{F}}$, where $V_1(p) := V(p) \cap |\mathfrak{G}_v|$. By construction of $\Delta_{\mathfrak{F}}$, we can show that $(\mathfrak{G}_v, V_1) \Vdash \Delta$ and $(\mathfrak{G}_v, V_1), v \Vdash p_X$ for any X with $w \in X \subseteq W$. Let us take some ω-saturated elementary extension $((\mathfrak{G}_v)^*, V_1^*)$ of $(\mathfrak{G}_v, V_1)$. Let v^* be the element in $(\mathfrak{G}_v)^*$ corresponding to v. Then, we obtain $((\mathfrak{G}_v)^*, V_1^*), v^* \Vdash \Delta_{\mathfrak{F}}$. We can also establish that $((\mathfrak{G}_v)^*, V_1^*) \Vdash \Delta$ and $((\mathfrak{G}_v)^*, V_1^*), v^* \Vdash p_X$ for any X with $w \in X \subseteq W$.

Let us define a mapping $f : |(\mathfrak{G}_v)^*| \to \mathrm{Uf}(W)$ by

$$f(s) := \{\, X \subseteq W \,:\, ((\mathfrak{G}_v)^*, V_1^*), s \Vdash p_X \,\}\,.$$

Now we are going to show the following: (a) $f(s)$ is an ultrafilter; (b) f is surjective; (c) f is a Fine mapping. Of course, the most important step for our purpose is to establish (c). So, we concentrate on showing (c). For the proof of (a) and (b), the reader can refer to [24]. Let us denote the accessibility relation of $(\mathfrak{G}_v)^*$ by S. We establish (**Forth**)- and (**Back**)-conditions for a Fine mapping (recall Definition 5.3). Consider any $s \in |(\mathfrak{G}_v)^*|$ and any $\mathcal{V} \in \mathrm{Uf}(W)$.

First, let us show (**Forth**). Assume that $\#\{\, s' \,:\, f(s') = \mathcal{V} \text{ and } sSs' \,\} \geq k$. We want to show $f(s) R_k^{\mathfrak{ue}} \mathcal{V}$. So, consider any $X \in \mathcal{V}$. We show $m_k(X) \in f(s)$, or equivalently, $((\mathfrak{G}_v)^*, V_1^*), s \Vdash p_{m_k(X)}$. Since $f(s') = \mathcal{V}$ and $X \in \mathcal{V}$ implies $X \in f(s')$, we obtain: $\{\, s' \,:\, f(s') = \mathcal{V} \text{ and } sSs' \,\} \subseteq \{\, s' \,:\, X \in f(s) \text{ and } sSs' \,\}$. By our assumption, $\#\{\, s' \,:\, X \in f(s) \text{ and } sSs' \,\} \geq k$. This gives us

$$\#\{\, s' \,:\, ((\mathfrak{G}_v)^*, V_1^*), s \Vdash p_X \text{ and } sSs' \,\} \geq k.$$

Thus $((\mathfrak{G}_v)^*, V_1^*), s \Vdash \Diamond_k p_X$. Since Δ is valid on $((\mathfrak{G}_v)^*, V_1^*)$, we can deduce that $((\mathfrak{G}_v)^*, V_1^*), s \Vdash p_{m_k(X)}$, as desired. We have shown (**Forth**).

Next, let us establish (**Back**). In what follows, we assume for simplicity that $k = 3$. Our assumption is $f(s) R_3^{\mathfrak{ue}} \mathcal{V}$. We show that $\#\{\, s' \,:\, f(s') = \mathcal{V} \text{ and } sSs' \,\} \geq 3$. Define

$$\begin{aligned}\Gamma :=& \{\, \mathbf{R}(\underline{s}, z_1) \wedge \mathbf{R}(\underline{s}, z_2) \wedge \mathbf{R}(\underline{s}, z_3) \wedge z_1 \neq z_2 \wedge z_1 \neq z_3 \wedge z_2 \neq z_3 \,\} \\ & \cup \{\, \mathbf{P}_Y(z_1) \wedge \mathbf{P}_Y(z_2) \wedge \mathbf{P}_Y(z_3) \,:\, Y \in \mathcal{V} \,\}\,,\end{aligned}$$

where $\mathbf{R}(x, y)$ is the binary symbol corresponding to S and each $\mathbf{P}_Y(x)$ is the unary predicate symbol corresponding to p_Y. By ω-saturation of $((\mathfrak{G}_v)^*, V_1^*)$, it suffices to show that Γ is finitely satisfiable. Consider $Y_1, \ldots, Y_n \in \mathcal{V}$. We show:

$$\begin{aligned}\Gamma' :=& \{\, \mathbf{R}(\underline{s}, z_1) \wedge \mathbf{R}(\underline{s}, z_2) \wedge \mathbf{R}(\underline{s}, z_3) \wedge z_1 \neq z_2 \wedge z_1 \neq z_3 \wedge z_2 \neq z_3 \,\} \\ & \cup \{\, \mathbf{P}_{Y_i}(z_1) \wedge \mathbf{P}_{Y_i}(z_2) \wedge \mathbf{P}_{Y_i}(z_3) \,:\, 1 \leq i \leq n \,\}\end{aligned}$$

is satisfiable in $((\mathfrak{G}_v)^*, V_1^*)$. Clearly, $Y_1 \cap \cdots \cap Y_n \in \mathcal{V}$. From $f(s) R_k^{\mathfrak{ue}} \mathcal{V}$ we have: $m_3(Y_1 \cap \cdots \cap Y_n) \in f(s)$, i.e., $((\mathfrak{G}_v)^*, V_1^*), s \Vdash p_{m_3(Y_1 \cap \cdots \cap Y_n)}$. Since Δ is valid on $((\mathfrak{G}_v)^*, V_1^*)$, we can deduce that $((\mathfrak{G}_v)^*, V_1^*), s \Vdash \Diamond_3(p_{Y_1} \wedge \cdots \wedge p_{Y_n})$, i.e., $\#(S(s) \cap [\![p_{Y_1} \wedge \cdots \wedge p_{Y_n}]\!]) \geq 3$. This means that Γ' is satisfiable. So we can conclude that Γ is finitely satisfiable in $((\mathfrak{G}_v)^*, V_1^*)$. We can easily generalize the above argument to other cases, where $k \neq 3$.

Therefore we have established the desired statement. □

Theorem 6.3 *An elementary class* F *of frames is definable by a set of formulas in GML* iff F *is closed under taking* (i) *generated subframes,* (ii) *disjoint unions,* (iii) *g-bounded morphic images, and* (iv) *graded ultrafilter images.*

Proof. The left-to-right direction is easy to show by Propositions 3.3, 3.10, 3.12 and Theorem 5.21. Conversely, assume that F satisfies the closure properties. Define $\mathrm{Log}(\mathsf{F}) := \{ \varphi : \mathsf{F} \Vdash \varphi \}$. We show that, for any $\mathfrak{F} \in \mathsf{F}$, $\mathfrak{F} \in \mathsf{F}$ iff $\mathfrak{F} \Vdash \mathrm{Log}(\mathsf{F})$. Consider $\mathfrak{F} \in \mathsf{F}$. It is trivial to show the Only-If-direction. Let us show the If-direction. Assume that $\mathfrak{F} \Vdash \mathrm{Log}(\mathsf{F})$. Similarly to the proof of Theorem 4.3, by our assumptions (i), (ii), (iii) and Proposition 3.13, we can assume without any loss of generality that $\mathfrak{F}$ is point-generated by a root $w \in |\mathfrak{F}|$.

Construct the 'complete description' $\Delta_{\mathfrak{F}}$ of $\mathfrak{F}$ (see Definition 6.1) [3]. Then we show that $\Delta_{\mathfrak{F}}$ is satisfiable in F as follows. It suffices to show that $\Delta_{\mathfrak{F}}$ is finitely satisfiable in F, because F is elementary. Let Γ be a finite subset of $\Delta_{\mathfrak{F}}$. To get a contradiction, suppose $\mathsf{F} \Vdash \neg \bigwedge \Gamma$. Then $\neg \bigwedge \Gamma \in \mathrm{Log}(\mathsf{F})$. Since $\mathfrak{F} \Vdash \mathrm{Log}(\mathsf{F})$, $\mathfrak{F} \Vdash \neg \bigwedge \Gamma$. However, $\bigwedge \Gamma$ is clearly satisfiable in $\mathfrak{F}$ under the natural valuation, which implies a contradiction. Therefore, $\Delta_{\mathfrak{F}}$ is satisfiable in some $\mathfrak{G} \in \mathsf{F}$.

By Lemma 6.2, there exists some $v \in |\mathfrak{G}|$ and some elementary extension $(\mathfrak{G}_v)^*$of $\mathfrak{G}_v$ such that $\mathfrak{F}$ is a graded ultrafilter image of $(\mathfrak{G}_v)^*$. By our closure properties and $\mathfrak{G} \in \mathsf{F}$, we can conclude that $\mathfrak{F} \in \mathsf{F}$ as required. □

In our proof of Theorem 6.3 (esp. Lemma 6.2), we require that a class F of frames is elementary. We essentially use a compactness argument that requires F to be elementary. So we can replace the assumption that F is elementary with closure under ultraproducts. Moreover, we can also reduce this condition to closure under ultrapowers as in the case of basic modal language as follows (cf. [13, Proposition 85]): an ultraproduct of frames is isomorphic to a generated subframe of the ultrapower, with respect to the same ultrafilter, of the disjoint union of the original frames. Since any graded modally definable class is closed under taking disjoint unions, we can apply the same argument to GML.

In basic modal language, we can also get even weaker closure condition: closure under ultrafilter extensions. Closure under ultrafilter extensions implies closure under ultraproducts, provided the intended class F is also closed under disjoint unions, generated subframes and isomorphisms. Thus, we can regard closure under ultrafilter extensions as the essential assumption of F in the proof of Goldblatt-Thomason Theorem in basic

[3] Remark that we can still assume that $\mathfrak{F} \Vdash \mathrm{Log}(\mathsf{F})$ in spite of our expansion of the original language in Definition 6.1. This is intuitively clear, because the choice of Prop is irrelevant, whenever we consider *frame validity.*

modal language. Then, the next question is: can we get a similar kind of closure condition even in GML? As we explained in the introduction, however, it seems difficult to find an appropriate notion of ultrafilter extension for GML in Kripke semantics. One possible way to avoid this difficulty is to change the semantics into the coalgebraic one [4].

7 Further Directions

7.1 The Scope of Our Guiding Idea

The first author and Sato [18,17] observed that there is a strong connection between the (strong) completeness proof of an extended modal logic and Goldblatt-Thomason characterization for it and they extracted an essential part of this connection, the notion of *realizer* [18]. The result of this paper gives us a further evidence of this observation. We can regard the notion of *Fine mapping* as the corresponding notion of realizer. Then our next question is: what is the scope of this observation? For example, we have not obtained a well-established model-theoretical study of conditional logic for preference frames [22]. Is it possible to apply the idea of [18] to obtain Goldblatt-Thomason-style results for conditional logic over preference frames?

7.2 Extensions of GML

We may consider extensions of GML with more expressive power. One general problem is definability of frame classes in extended GMLs. Lethinen [16] studied the extension of GML obtained by adding the path quantifier, which allows us to talk about the truth on all paths starting from a certain state, the modal μ-extension, and an infinite GML. Some excellent results on relative definability of frame classes are proved there. Another nice extension is graded hybrid logic (GHL). Kaminski, et. al. [14] studied terminating tableaux for graded hybrid logic. However we still lack a Goldblatt-Thomason-style characterization for GHL. Is it possible to merge our idea from this paper with Ten Cate's Goldblatt-Thomason-style characterization for hybrid logic [20], in particular, his idea of *ultrafilter morphic images* [20, Definition 4.2.5]? This would be a promising further direction.

7.3 Coalgebraic GML

We have shown that there is an alternative semantics which interprets GML on directed graphs. Another excellent (but related) alternative semantics is the coalgebraic one. This was shown by D'Agostino and Visser [5] and it was claimed that graded modal logic is subsumed under coalgeraic modal logics. We may define a functor Ω on the category of sets such that $\Omega(X) = (\omega+1)^X$, the set of all functions from X to $\omega+1$. A Ω-coalgebra is a pair (X, σ) where $\sigma : X \to \Omega(X)$ is a transition map. Recently the second author has found out that there is a natural Goldblatt-Thomason theorem for

[4] Recently, the second author has found the notion of ultrafilter extension in the coalgebraic semantics and proved a natural Goldbatt-Thomason Theorem for GML (see also Section 7.3)

coalgebraic GML, by applying duality between algebras and Ω-coalgebras. Remark that this is not a consequence of the general Goldblatt-Thomason theorem given by Kurz and Rosický [15], because they restrict functors to those preserving finiteness, i.e., mapping finite sets to finite sets, while the functor Ω above does not preserve finiteness.

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A Proofs of Lemmas

The proof of Lemmas 5.16 and 5.17 adapts and extends the proof of Theorem 1 from [10].

A.1 Proof of Lemma 5.16

By induction on k.

(Base) Since $\#\mathrm{T}_{\mathcal{U}}(X) \geq 0$, there is nothing to prove in the case where $k = 0$. Consider the case where $k = 1$. Assume that $m_1(X) \in \mathcal{U}$. Since $m_1(X) := \{ w \in W : R(w) \cap X \neq \emptyset \}$, we can identify $(\mathrm{Uf}(W), R_1^{\mathfrak{ue}})$ with the ultrafilter extension of $\mathfrak{F}$ for the basic modal language. Then from $m_1(X) \in \mathcal{U}$ we easily deduce that there exists an ultrafilter $\mathcal{U}'$ such that $\mathcal{U}' \subseteq \{ Y \subseteq W : m_1(Y) \in \mathcal{U} \}$ and $X \in \mathcal{U}'$. So we have $\mathcal{U} R_1^{\mathfrak{ue}} \mathcal{U}'$. This yields us $(\mathcal{U}', 1) \in \mathrm{T}_{\mathcal{U}}(X)$, hence $\#\mathrm{T}_{\mathcal{U}}(X) \geq 1$.

(Induction Step) Consider the case where $k > 1$. First, note that our induction hypothesis is: $\forall\, l < k.\ \forall\, X \subseteq W.\ \ m_l(X) \in \mathcal{U}$ implies $\#\mathrm{T}_{\mathcal{U}}(X) \geq l$. Assume that $m_k(X) \in \mathcal{U}$. By Proposition 5.10 (iii), we have:

$$\forall\, Y \subseteq W.\ \exists\, i \leq k.\ m_i(X \cap Y) \cap m_{k-i}(X \cap (W \setminus Y)) \in \mathcal{U}. \qquad (*)$$

For simplicity, let us abbreviate $(*)$ as $\forall\, Y \subseteq W.\ \exists\, i \leq k.\ \Psi_{\mathcal{U}}(Y, i)$. We subdivide our argument into the following two cases:
(a) $\exists\, Y \subseteq W.\ \exists\, i \leq k.\ (\Psi_{\mathcal{U}}(Y, i)$ and $i \neq 0$ and $i \neq k)$;
(b) $\forall\, Y \subseteq W.\ \forall\, i \leq k.\ (\Psi_{\mathcal{U}}(Y, i)$ implies $(i = 0$ or $i = k))$.
First, let us consider the easier case (a). Fix Y and i with $\Psi_{\mathcal{U}}(Y, i)$. Then we have $1 < i < k$. So by our I.H. and $\Psi_{\mathcal{U}}(Y, i)$, we can state that: $\#\mathrm{T}_{\mathcal{U}}(X \cap Y) \geq i$ and

$\#\mathrm{T}_{\mathcal{U}}(X \cap (W \setminus Y)) \geq k - i$. By Proposition 5.15, we can calculate as follows:

$$\#\mathrm{T}_{\mathcal{U}}(X) = \#\mathrm{T}_{\mathcal{U}}(X \cap Y) + \#\mathrm{T}_{\mathcal{U}}(X \cap (W \setminus Y)) \geq i + (k - i) = k,$$

as desired. This finishes the case (a).

Let us consider the case (b). First, we show the following claim:

Claim A.1 *For any $Y \subseteq W$, $l_R(X \Rightarrow (W \setminus Y)) \cup l_R(X \Rightarrow Y) \in \mathcal{U}$.*

(Proof of Claim) Let us consider any $Y \subseteq W$. By our assumption ($*$), we can find $j \leq k$ such that $\Psi_{\mathcal{U}}(Y, j)$, i.e., $m_j(X \cap Y) \in \mathcal{U}$ and $m_{k-j}(X \cap (W \setminus Y)) \in \mathcal{U}$. Then, (b) teaches us that we have $j = 0$ or $j = k$. Here we only check the desired conclusion in the case $j = 0$, because we can similarly show it in the case $j = k$. Let us assume that $j = 0$. Then we have $m_k(X \cap (W \setminus Y)) \in \mathcal{U}$ (note: $m_0(X \cap Y) \in \mathcal{U}$ always holds). We are going to establish $l_R(X \Rightarrow (W \setminus Y)) \in \mathcal{U}$, i.e., $W \setminus m_R(X \cap Y) \in \mathcal{U}$. So suppose that on the contrary, $m_R(X \cap Y) \in \mathcal{U}$, hence $m_1(X \cap Y) \in \mathcal{U}$. But by Proposition 5.10 (ii) and $m_k(X \cap (W \setminus Y)) \in \mathcal{U}$, we obtain $m_{k-1}(X \cap (W \setminus Y)) \in \mathcal{U}$ (note that $k > 1$). Together with $m_1(X \cap Y) \in \mathcal{U}$, this means that we have shown $\Psi_{\mathcal{U}}(Y, 1)$. Then from (b) it follows that $1 = 0$ or $1 = k$. Both of the disjuncts, however, are impossible. So we get the desired contradiction. Therefore, $l_R(X \Rightarrow (W \setminus Y)) \in \mathcal{U}$ hence $l_R(X \Rightarrow (W \setminus Y)) \cup l_R(X \Rightarrow Y) \in \mathcal{U}$. ⊣

By this claim, we can also establish the following.

Claim A.2 $\{X\} \cup \{Y \subseteq W : W \setminus m_k(W \setminus Y) \in \mathcal{U}\}$ *satisfies the finite intersection property.*

(Proof of Claim) Suppose the contrary — that this set does not have the finite intersection property. It follows that there exists finite $Y_1, \ldots Y_n$ such that $W \setminus m_k(W \setminus Y_i) \in \mathcal{U}$ $(1 \leq i \leq n)$ and $X \cap Y_1 \cap \cdots \cap Y_n = \emptyset$. We subdivide our argument into the following two cases:

(i) $\forall i. (1 \leq i \leq n$ implies $l_R(X \Rightarrow Y_i) \in \mathcal{U})$;

(ii) $\exists i. (1 \leq i \leq n$ and $l_R(X \Rightarrow Y_i) \notin \mathcal{U})$.

First, let us consider the case (i). Then from (i) we derive that $l_R(X \Rightarrow (Y_1 \cap \cdots \cap Y_n)) \in \mathcal{U}$. By our assumption $X \cap Y_1 \cap \cdots \cap Y_n = \emptyset$ (i.e., $Y_1 \cap \cdots \cap Y_n \subseteq (W \setminus X)$), however, we have: $l_R((Y_1 \cap \cdots \cap Y_n) \Rightarrow (W \setminus X)) \in \mathcal{U}$. Thus we obtain $l_R(W \setminus X) \in \mathcal{U}$ hence $W \setminus m_R(X) \in \mathcal{U}$. Recall that our assumption for the induction step is $m_k(X) \in \mathcal{U}$, which implies a contradiction by Proposition 5.10 (ii). This finishes the case (i).

Finally, consider the case (ii). Fix i with $l_R(X \Rightarrow Y_i) \notin \mathcal{U}$. By Claim A.1, $l_R(X \Rightarrow (W \setminus Y_i)) \in \mathcal{U}$. Recall that Y_i satisfies $W \setminus m_k(W \setminus Y_i) \in \mathcal{U}$. Then by Proposition 5.10 (iv), we can establish that $W \setminus m_k(X) \in \mathcal{U}$, which gives us a contradiction to our assumption: $m_k(X) \in \mathcal{U}$. This finishes the case (ii). ⊣

By Claim A.2, we can find an ultrafilter $\mathcal{U}'$ such that $X \in \mathcal{U}'$ and $Y \in \mathcal{U}'$ for any Y with $W \setminus m_k(W \setminus Y) \in \mathcal{U}$. Thus, we have $\mathcal{U} R_k^{\mathfrak{ue}} \mathcal{U}'$ and $X \in \mathcal{U}'$. From Proposition 5.10 (ii) it follows that $\mathcal{U} R_l^{\mathfrak{ue}} \mathcal{U}'$ for any l with $1 \leq l \leq k$, i.e., $(\mathcal{U}', l) \in \mathrm{T}_{\mathcal{U}}(X)$ for any l with $1 \leq l \leq k$. Therefore we have shown $\#\mathrm{T}_{\mathcal{U}}(X) \geq k$, as required. This completes our proof for the induction step. □

A.2 Proof of Lemma 5.17

By induction on k.

(Base) Since $m_0(X) \in \mathcal{U}$ always holds, there is nothing to prove in the case where $k = 0$. Consider the case $k = 1$ and assume that $l > 1$ and $\#\mathrm{T}_{\mathcal{U}}(X) \geq 1$. So we can find $(\mathcal{U}', l) \in \mathrm{T}_{\mathcal{U}}(X)$. Equivalently, $\mathcal{U}R_l^{\mathfrak{ue}}\mathcal{U}'$ and $X \in \mathcal{U}'$. From Proposition 5.10 (ii) we can derive that $\mathcal{U}R_1^{\mathfrak{ue}}\mathcal{U}'$ and $X \in \mathcal{U}'$ hence $m_1(X) \in \mathcal{U}$, as required.

(Induction Step) Consider the case $k > 1$. Assume that $\#\mathrm{T}_{\mathcal{U}}(X) \geq k$. We split our argument into the following cases:

(a) $\exists\, (\mathcal{U}_1, l_1) \in \mathrm{T}_{\mathcal{U}}(X).\ \exists\, (\mathcal{U}_2, l_2) \in \mathrm{T}_{\mathcal{U}}(X).\, (\mathcal{U}_1 \neq \mathcal{U}_2)$.

(b) $\forall\, (\mathcal{U}_1, l_1) \in \mathrm{T}_{\mathcal{U}}(X).\ \forall\, (\mathcal{U}_2, l_2) \in \mathrm{T}_{\mathcal{U}}(X).\, (\mathcal{U}_1 = \mathcal{U}_2)$.

First, we consider the case (b). Our assumption $\#\mathrm{T}_{\mathcal{U}}(X) \geq k$ and (b) allow us to establish $(\mathcal{U}_1, l_1) \in \mathrm{T}_{\mathcal{U}}(X)$ for some ultrafilter $\mathcal{U}_1$ and some $l_1 \geq k$. Fix such $\mathcal{U}_1$ and $l_1 \geq k$. Since $\mathcal{U}R_{l_1}^{\mathfrak{ue}}\mathcal{U}_1$ and $X \in \mathcal{U}_1$, from Proposition 5.10 (ii) we deduce that $\mathcal{U}R_k^{\mathfrak{ue}}\mathcal{U}_1$ and $X \in \mathcal{U}_1$ hence $m_k(X) \in \mathcal{U}$, as desired.

Second, let us consider the case (a). Take some $(\mathcal{U}_1, l_1)$, $(\mathcal{U}_2, l_2) \in \mathrm{T}_{\mathcal{U}}(X)$ with $\mathcal{U}_1 \neq \mathcal{U}_2$. From $\mathcal{U}_1 \neq \mathcal{U}_2$ it follows that $Y \in \mathcal{U}_1$, but $W \setminus Y \in \mathcal{U}_2$ for some $Y \subseteq W$. Since $(\mathcal{U}_1, l_1)$, $(\mathcal{U}_2, l_2) \in \mathrm{T}_{\mathcal{U}}(X)$, we can state that: (i) $\mathcal{U}R_{l_1}^{\mathfrak{ue}}\mathcal{U}_1$ and $X \cap Y \in \mathcal{U}_1$; (ii) $\mathcal{U}R_{l_2}^{\mathfrak{ue}}\mathcal{U}_2$ and $X \cap (W \setminus Y) \in \mathcal{U}_2$. So $\mathrm{T}_{\mathcal{U}}(X \cap Y) \neq \emptyset$ and $\mathrm{T}_{\mathcal{U}}(X \cap (W \setminus Y)) \neq \emptyset$. Hence by Proposition 5.15 (ii), we can establish the following:

$$\exists\, i.\, (0 < i < k \text{ and } \#\mathrm{T}_{\mathcal{U}}(X \cap Y) \geq i \text{ and } \#\mathrm{T}_{\mathcal{U}}(X \cap (W \setminus Y)) \geq k - i).$$

Then from I.H. it follows that $m_i(X \cap Y) \in \mathcal{U}$ and $m_{k-i}(X \cap (W \setminus Y)) \in \mathcal{U}$. Therefore, by Proposition 5.10 (iii), $m_k(X) \in \mathcal{U}$, as required. This finishes our proof of the induction step. □