Uniform Interpolation for Monotone Modal Logic

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Abstract

We reconstruct the syntax and semantics of monotone modal logic, in the style of Moss' coalgebraic logic. To that aim, we replace the box and diamond with a modality ∇ which takes a finite collection of finite sets of formulas as its argument. The semantics of this modality in monotone neighborhood models is defined in terms of a version of relation lifting that is appropriate for this setting.
We prove that the standard modal language and our ∇-based one are effectively equi-expressive, meaning that there are effective translations in both directions. We prove and discuss some algebraic laws that govern the interaction of ∇ with the Boolean operations. These laws enable us to rewrite each formula into a special kind of disjunctive normal form that we call transparent. For such transparent formulas it is relatively easy to define the bisimulation quantifiers that one may associate with our notion of relation lifting. This allows us to prove the main result of the paper, viz., that monotone modal logic enjoys the property of uniform interpolation.

Keywords: monotone modal logic, uniform interpolation, neighborhood semantics, coalgebra

1 Introduction

Monotone modal logic is a generalization of normal modal logic in which the distribution of $\Box$ over conjunctions has been weakened to a monotonicity condition, which can either be expressed as an axiom $((\Box(p \land q) \to \Box p)$, or as a rule (from $p \to q$ derive $\Box p \to \Box q$). The standard semantics for such logics is provided by so-called monotone neighborhood models. Here the binary relation over a state space S is generalized to a so-called *monotone neighborhood function*, that is, a map $\sigma : S \to \mathcal{PP}S$ which is closed under taking supersets (if $X \in \sigma(s)$ and $X \subseteq Y$ then $Y \in \sigma(s)$). The interpretation of the modality in a monotone neighborhood model $\mathbb{S} = \langle S, \sigma, V \rangle$ (with V a valuation) is then given by

$$\mathbb{S}, s \Vdash \Box a \iff \exists U \in \sigma(s)\ \forall u \in U.\ \mathbb{S}, u \Vdash a\,.$$

Together with its polymodal versions and its fixed-point extensions, this logic has found applications in settings where the use of normal modalities would have undesirable consequences, such as in deontic [7], epistemic [35], or game logic [27]. The latter application, in which the monotone neighborhood function encodes the power of a player to achieve a certain outcome in an interactive system or game, has received some attention in computer science lately [1,28]. Monotone modal logic also appears to be a crucial link between lattice theory and modal logic, through the new covering semantics of lattices and substructural logics [15,31].

Although the monotone variant has never taken a central place in modal logic, various technical results are known. Two sources of information are the textbook [7], and the more recent survey [16], which also contains many original results. We mention here some facts concerning **M**, the monotone variant of the basic modal logic **K**. **M** is sound and complete with respect to the neighborhood semantics and satisfies the finite model property [7]; its satisfiability problem is NP-complete [35]. Finally, **M** has the Craig interpolation property [16]: Given two modal formulas a, b, if $\models a \to b$ (meaning that $a \to b$ holds in every state of every monotone neighborhood model), then there is a formula c, which may only use propositional variables that appear both in a and in b, such that $\models a \to c$ and $\models c \to b$.

The main contribution of this paper will be to add *uniform* interpolation to the list of properties of monotone modal logic. Uniform interpolation is a very strong version of interpolation, where the interpolant c does not really depend on b itself, but only on the *language* it shares with a (that is, the set of variables occurring both in a and in b). More precisely, we shall prove the following result. Here P_a denotes the set of proposition letters occurring in a given formula a, and $\mathcal{L}_\Diamond(\mathsf{Q})$ denotes the set of modal formulas a with $\mathsf{P}_a \subseteq \mathsf{Q}$.

Theorem 1 (Uniform interpolation for M) *For any modal formula a and any set $\mathsf{Q} \subseteq \mathsf{P}_a$ of proposition letters, there is a formula $a_\mathsf{Q} \in \mathcal{L}_\Diamond(\mathsf{Q})$, effectively constructible from a, such that for any formula b with $\mathsf{P}_a \cap \mathsf{P}_b \subseteq \mathsf{Q}$ we have*

$$\models a \to b \quad \textit{iff} \quad \models a_\mathsf{Q} \to b. \tag{1}$$

Observe that by (1) it follows from $\models a_\mathsf{Q} \to a_\mathsf{Q}$ and $a_\mathsf{Q} \in \mathcal{L}_\Diamond(\mathsf{Q})$ that $\models a \to a_\mathsf{Q}$, and so a_Q is indeed an interpolant for every b with $\mathsf{P}_a \cap \mathsf{P}_b \subseteq \mathsf{Q}$: if $\models a \to b$ then $\models a \to a_\mathsf{Q}$ and $\models a_\mathsf{Q} \to b$.

A survey on (uniform) interpolation and the tools used to prove this property appears in [9, Section 4]. While it is easily argued that classical propositional logic has uniform interpolation, not many logics have this property, for example, first-order logic lacks it [19]. Recent interest in the property was initiated by the seminal work by A. Pitts [29] who proved that intuitionistic logic has the uniform interpolation property. In modal logic, Shavrukov [33] independently proved that the provability logic (also known as Gödel-Löb logic) **GL** has uniform interpolation. Subsequently, the property was established for modal logic **K**, independently by Ghilardi [13] and Visser [36], while [14] contains negative results on modal logics like **S4**. Finally, in the theory of modal fixed-point logics, it was realized in [10] that the logical property of uniform interpolation corresponds to the automata-theoretic property of *closure under projection*. In the same paper it was proved that the full modal μ-calculus [21] has uniform interpolation, in contrast to the fact that for instance PDL lacks the property [23].

Proofs of uniform interpolation property either follow a proof-theoretic or a semantic road. Notable examples of the proof-theoretic approach are Pitts' work [29] and, for modal logics, [5]. The semantic approach towards proving uniform interpolation is based on proving that a certain nonstandard second-order quantifier is definable in the language [29,36]. This *bisimulation quantifier* is interpreted as follows:

$$\mathbb{S}, s \Vdash \exists p.a \;\text{ iff }\; \mathbb{S}', s' \Vdash a, \;\text{ for some } \mathbb{S}', s' \text{ with } \mathbb{S}, s \simeq_p \mathbb{S}', s', \tag{2}$$

where $\simeq_p$ denotes the relation of bisimilarity *up to proposition letter* p (see Definition 2.5 for a precise definition). Intuitively, (2) says that we can make the formula a true by, indeed, changing the interpretation of p, although not necessarily here, but in an up-to-p bisimilar state. For an detailed study of bisimulation quantifiers in modal logic, see [12].

In the case of the normal modal logic **K**, the proof simplifies considerably if we reconstruct the modal language on the basis of the so-called *cover modality*, here written as $\nabla_{\mathcal{P}}$. This modality, which takes a finite set of formulas as its argument, was introduced as a primitive operator, independently by Barwise & Moss [4] and by Janin & Walukiewicz [20]. Moss [25] observed that the semantics of this modality, which takes a *set* of formulas as its argument, can be defined in terms of *relation lifting*. More precisely, given a relation $R \subseteq S \times S'$, define

$$\begin{aligned}
\overrightarrow{\mathcal{P}}(R) &:= \{ (A, A') \in \mathcal{P}(S) \times \mathcal{P}(S') \mid \forall a \in A\, \exists a' \in A'.(a, a') \in R \},\\
\overleftarrow{\mathcal{P}}(R) &:= \{ (A, A') \in \mathcal{P}(S) \times \mathcal{P}(S') \mid \forall a' \in A'\, \exists a \in A.(a, a') \in R \},\\
\overline{\mathcal{P}}(R) &:= \overrightarrow{\mathcal{P}}(R) \cap \overleftarrow{\mathcal{P}}(R).
\end{aligned}$$

The relation $\overline{\mathcal{P}}(R)$ is called the Egli-Milner lifting of R — note that this relation underlies the back-and-forth clauses in the definition of a bisimulation between Kripke models. Returning to the semantics of $\nabla_{\mathcal{P}}$, given a Kripke model $\mathbb{S}$, we may consider the Egli-Milner lifting $\overline{\mathcal{P}}(\Vdash) \subseteq \mathcal{P}(S) \times \mathcal{P}(\mathcal{L})$ of the satisfaction relation $\Vdash \subseteq S \times \mathcal{L}$ between states and formulas, and define:

$$\mathbb{S}, s \Vdash \nabla_{\mathcal{P}} A \;\text{ iff }\; (\rho(s), A) \in \overline{\mathcal{P}}(\Vdash), \tag{3}$$

where $\rho(s)$ is the set of successors of s. Recently, axiomatic bases and proof systems have been found for languages based on $\nabla_{\mathcal{P}}$ and its generalization [6,22].

Two properties make the cover modality $\nabla_{\mathcal{P}}$ very useful: First, the connectives $\nabla_{\mathcal{P}}$ and $\vee$ have in some sense the same expressive power as the set $\{ \vee, \wedge, \Diamond, \Box \}$. And second, the bisimulation quantifier distributes both over $\vee$ and over $\nabla_{\mathcal{P}}$:

$$\exists p.\nabla_{\mathcal{P}} A \equiv \nabla_{\mathcal{P}} \{ \exists p.a \mid a \in A \}.$$

As a consequence, once we have 'reconstructed' the modal language on the basis of $\vee$ and $\nabla_{\mathcal{P}}$, the bisimulation quantifiers can be defined by a trivial inductive definition. This approach to uniform interpolation goes back to [10,11]. In [34] an algorithm is given computing uniform interpolants of size (singly) exponential in the size of the original formula.

We shall mimick this approach in our proof of Theorem 1. There is a natural notion of bisimilarity associated with monotone neighborhood models, and so we may naturally interpret the bisimulation quantifiers along this relation. Monotone bisimilarity can be expressed in terms of a relation lifting as well [18]. This lifting $\widetilde{\mathcal{M}}$ is given by defining, for a relation $R \subseteq S \times S'$, the relation $\widetilde{\mathcal{M}}(R) \subseteq \mathcal{P}\mathcal{P}(S) \times \mathcal{P}\mathcal{P}(S')$ as follows:

$$\widetilde{\mathcal{M}}(R) := \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}(R) \cap \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}(R).$$

Our idea is now to introduce a modality ∇ for monotone modal logic, which, analogous to the cover modality for relational models, is interpreted in neighborhood models by means of the lifting $\widetilde{\mathcal{M}}(\Vdash)$ of the satisfaction relation $\Vdash$.

Taking this idea as our guideline, we arrive at the following 'reconstruction' of monotone modal logic. Our language $\mathcal{L}_\nabla$ is based on a nonstandard modality ∇ which takes finite collections of finite sets of formulas as its argument:

- $\nabla\alpha$ is a formula of $\mathcal{L}_\nabla$, for each $\alpha \in \mathcal{P}_\omega\mathcal{P}_\omega\mathcal{L}_\nabla$.

The semantics of this operator is expressed in terms of the relation lifting $\widetilde{\mathcal{M}}$. That is, in every monotone neighborhood model $\mathbb{S}$, we have

$$\mathbb{S}, s \Vdash \nabla\alpha \text{ iff } (\sigma(s), \alpha) \in \widetilde{\mathcal{M}}(\Vdash). \tag{4}$$

The main aim of this paper is to show that this alternative way to set up monotone modal logic makes sense: With some work we can prove results analogous to the relational case. To start with, Theorem 3.5 below states that the standard modal language and our ∇-based one are effectively equi-expressive, meaning that there are effective translations in both directions. We prove and discuss some algebraic laws that guide the interaction of ∇ with the Boolean operations. These laws may not be as straightforward as in the relational case, but still they enable us to rewrite each formula into a special kind of disjunctive normal form that we call transparent (Proposition 5.3). For such transparent formulas it is relatively easy to define the bisimulation quantifiers associated with our notion $\widetilde{\mathcal{M}}$ of relation lifting (Definition 5.4). This allows us to prove the main result of the paper, viz., Theorem 1 above.

Finally, our approach has been very much influenced by the *coalgebraic* perspective on modal logic. The theory of (Universal) Coalgebra [30] provides a general mathematical framework for studying behavior of state-based evolving systems. Key examples of such systems are Kripke frames and models, together with many other structures from the theory of modal logic, such as (weighted/probabilistic) transition systems, general frames, and neighborhood structures. The link between modal logic and coalgebra is in fact very tight: Modal logic, suitably generalized and modified, provides natural languages and derivation systems for specifying and reasoning about behavior at a coalgebraic level of generality [8]. For a coalgebraic perspective on monotone modal logic the reader is referred to [17].

The link between modal logic and coalgebra goes back to the work of Moss [25], who observed that modal logic, once formulated in terms of the cover modality $\nabla_{\mathcal{P}}$, can be generalized to coalgebras of arbitrary type. Each coalgebraic type T, formally given as a functor $T : \mathsf{Set} \to \mathsf{Set}$, canonically induces an operation $\overline{T}$, which lifts a relation $R \subseteq S \times S'$ to a relation $\overline{T}(R) \subseteq TS \times TS'$. On this basis, Moss develops a modal formalism, based on a modality ∇_T, of which the semantics is defined in terms of the lifting $\overline{T}(\Vdash)$. Our modality ∇ is inspired by Moss' approach, instantiated by the type $\mathcal{M}$ of neighborhood frames. However, our notion of relation lifting, $\widetilde{\mathcal{M}}$, *differs* from the canonically defined relation lifting, $\overline{\mathcal{M}}$. We return to this issue in the final section of the paper.

Overview In the next section we recall some definitions, introduce some basic notions, and discuss some new concepts, including some properties of our relation lifting. In section 3 we introduce the monotone nabla modality, and we prove the equi-expressiveness result. In section 4 we discuss some algebraic laws that govern the interaction of ∇ with the Boolean connectives. Section 5 is the main part of the paper: here we show how to define bisimulation quantifiers in the language $\mathcal{L}_\nabla$, and we show how to derive uniform interpolation from that. We finish with drawing some conclusions and listing some future work.

2 Preliminaries

2.1 Neighborhood models and monotone modal logic

We first recall some basic facts on monotone modal logic. It will be convenient for us to base our language on formulas in negation normal form, in which the use of negations is restricted to atomic formulas. As a consequence, all our primitive connectives will come in pairs of Boolean duals. In particular, next to the modality $\Box$ we also have to take its Boolean dual, $\Diamond$, as a primitive connective.

Definition 2.1 Given a set Prop of proposition letters, the set $\mathcal{L}_\Diamond(\mathsf{Prop})$ of *modal formulas* over Prop, is given by the following grammar:

$$a \;:=\; p \mid \neg p \mid \bot \mid \top \mid a \wedge a \mid a \vee a \mid \Diamond a \mid \Box a$$

where $p \in \mathsf{Prop}$.

Definition 2.2 A *neighborhood frame* is a pair $\langle S, \sigma\rangle$ such that $\sigma : S \to \mathcal{P}\mathcal{P}(S)$ is a map assigning to a state $s \in S$ a collection $\sigma(s)$ of *neighborhoods* of s. In case each $\sigma(s)$ is closed under taking supersets (that is, if $Y \supseteq X \in \sigma(s)$ implies $Y \in \sigma(s)$), we say that the neighborhood frame is *monotone.* A *neighborhood model* is a triple $\mathbb{S} = \langle S, \sigma, V\rangle$ such that $\langle S, \sigma\rangle$ is a neighborhood frame, and $V : S \to \mathcal{P}(\mathsf{Prop})$ is a *coloring.* Neighborhood models based on monotone frames will simply be called *models.* A *pointed model* is just a pair $(\mathbb{S}, s)$ consisting of a model $\mathbb{S}$ and a point s in $\mathbb{S}$.

Definition 2.3 Given a model $\mathbb{S} = \langle S, \sigma, V \rangle$, we define the satisfaction relation $\Vdash \subseteq S \times \mathcal{L}_\diamond(\mathsf{Prop})$ by induction. For the classical connectives the definition is as usual:

$\mathbb{S}, s \Vdash p$ iff $p \in V(s)$	$\mathbb{S}, s \Vdash \neg p$ iff $p \notin V(s)$
$\mathbb{S}, s \Vdash \top$ iff `true`	$\mathbb{S}, s \Vdash \bot$ iff `false`
$\mathbb{S}, s \Vdash \phi \wedge \psi$ iff $\mathbb{S}, s \Vdash \phi$ and $\mathbb{S}, s \Vdash \psi$	$\mathbb{S}, s \Vdash \phi \vee \psi$ iff $\mathbb{S}, s \Vdash \phi$ or $\mathbb{S}, s \Vdash \psi$.

For the modal connectives we define:

$$\mathbb{S}, s \Vdash \Box\phi \text{ iff } \exists U \in \sigma(s) \text{ such that } \forall u \in U\ \mathbb{S}, u \Vdash \phi$$
$$\mathbb{S}, s \Vdash \Diamond\phi \text{ iff } \forall U \in \sigma(s)\ \exists u \in U \text{ such that } \mathbb{S}, u \Vdash \phi\,.$$

Convention 2.4 *In order to recall the logical structure of the satisfaction relation of the two modal connectors, we shall write* $\langle\exists\forall\rangle$ *in place of* $\Box$*, and* $\langle\forall\exists\rangle$ *in place of* $\Diamond$*.*

Definition 2.5 Let $\mathbb{S} = \langle S, \sigma, V \rangle$ and $\mathbb{S}' = \langle S', \sigma', V' \rangle$ be two models, and let $\mathsf{P} \subseteq \mathsf{Prop}$ be a set of proposition letters. A relation $Z \subseteq S \times S'$ is a P-*bisimulation*, if for all $(s, s') \in Z$:
(prop) $V(s) \cap \mathsf{P} = V'(s') \cap \mathsf{P}$;
(forth) $\forall U \in \sigma(s) \exists U' \in \sigma'(s') \forall u' \in U' \exists u \in U.\, uZu'$;
(back) $\forall U' \in \sigma'(s') \exists U \in \sigma(s) \forall u \in U \exists u' \in U'.\, uZu'$.
Prop-bisimulations are simply called *bisimulations*, and $(\mathsf{Prop} \setminus \{\, p \,\})$-bisimulations will be called *bisimulations up to* p. If Z is a P-bisimulation between $\mathbb{S}$ and $\mathbb{S}'$ linking s to s', we write $Z : \mathbb{S}, s \simeq_{\mathsf{P}} \mathbb{S}', s'$.

2.2 Functors and coalgebras

While we shall generally suppress the use of category theory in this paper, we need the set functors, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{M}$, and their finitary versions, $\mathcal{P}_\omega$, $\mathcal{Q}_\omega$ and $\mathcal{M}_\omega$. As mentioned, neighborhood frames are coalgebras for the functor $\mathcal{M}$.

Definition 2.6 On the category Set (with sets as objects and functions as arrows) we let $\mathcal{P}$ denote the covariant *power set functor*; $\mathcal{Q} := \mathcal{P} \circ \mathcal{P}$ is the *double power set functor*. We write $\mathcal{P}_\omega(S)$ for the collection of all finite subsets of S. As the direct image of a finite subset is finite, $\mathcal{P}_\omega$ is itself a functor. We define $\mathcal{Q}_\omega := \mathcal{P}_\omega \circ \mathcal{P}_\omega$.

Given an element $\alpha \in \mathcal{Q}S$, we define

$$\alpha^\uparrow := \{\, X \in \mathcal{P}(S) \mid X \supseteq Y \text{ for some } Y \in \alpha \,\},$$

and we say that α is upward closed if $\alpha = \alpha^\uparrow$. The functor $\mathcal{M}$ is given by $\mathcal{M}(S) := \{\, \alpha \in \mathcal{Q}S \mid \alpha \text{ is upward closed } \}$, while for $f : S \to S'$, we define $\mathcal{M}f : \mathcal{M}S \to \mathcal{M}S'$ by $(\mathcal{M}f)(\alpha) := ((\mathcal{Q}f)(\alpha))^\uparrow$. For $\mathcal{M}_\omega$ we define $\mathcal{M}_\omega(S) := \{\alpha \in \mathcal{M}(S) \mid \alpha = \beta^\uparrow \text{ for some } \beta \in \mathcal{Q}_\omega(S)\}$. It is easily verified that by putting $\mathcal{M}_\omega f(\alpha) := \mathcal{M}f(\alpha)$, we turn $\mathcal{M}_\omega$ itself into a functor.

Instead of working with an element $\alpha \in \mathcal{M}_\omega S$ it will often be convenient to work with a finite generating set, that is, a set $\beta \in \mathcal{Q}_\omega S$ such that $\alpha = \beta^\uparrow$. Fortunately there is always a canonical choice for such a β: we leave it for the reader to verify that the following is well-defined.

Definition 2.7 An element $\beta \in \mathcal{Q}S$ is an *anti-chain* if $X \subseteq Y$ for no $X, Y \in \beta$. Given a set $\alpha \in \mathcal{M}_\omega S$, we let $\alpha_\downarrow$ denote the unique anti-chain $\beta \in \mathcal{Q}_\omega S$ with $\alpha = \beta^\uparrow$.

2.3 The exchange operator

One key argument in our proofs shall be a principle of quantifier exchange, that we state in Proposition 2.12. This principle can be understood algebraically as providing a direct characterization of the closure operator arising from a Galois connection. Typically, such a closure operator is defined in terms of universal quantifiers, while the characterization we provide of the same operator is an existential statement.

Definition 2.8 Given some set S, for $A, B \subseteq \mathcal{P}(S)$, we write

$$A \perp_S B \text{ iff } A \cap B \neq \emptyset,$$

and for $\alpha \in \mathcal{Q}S$ we define

$$\alpha^{\perp_S} := \{ B \in \mathcal{P}(S) \mid B \perp_S A \text{ for all } A \in \alpha \}.$$

In lattice theory, the operation $(\,\cdot\,)^{\perp_S}$ is known as the *polarity* associated with the relation $\perp_S$, and as such it is well-known to have some nice properties. To ease the reading we shall omit the subscript S from $(\,\cdot\,)^{\perp_S}$, if no confusion is likely to arise. The following observation is straightforward.

Proposition 2.9 *Given a set S, the operation $(\,\cdot\,)^{\perp\perp}$ is a closure operation on $\mathcal{Q}S$, with $\mathcal{M}S \subseteq \mathcal{Q}S$ forming the set of closed elements. For $\alpha \in \mathcal{Q}S$, we have*
(1) $(\alpha^\perp)^\perp = \alpha^\uparrow$;
(2) $\alpha^\perp = (\alpha^\uparrow)^\perp$;
(3) $\alpha = \varnothing$ iff $\varnothing \in \alpha^\perp$;
(4) $\varnothing \in \alpha$ iff $\alpha^\perp = \varnothing$.

Given $\alpha \in \mathcal{Q}_\omega S$, the set $\alpha^{\perp_S} \in \mathcal{Q}S$ need not belong to $\mathcal{Q}_\omega S$ (unless S is finite), but fortunately we can make the following observations.

Proposition 2.10 *Given some set S, assume that $\alpha \in \mathcal{M}_\omega S \cup \mathcal{Q}_\omega S$. Then*
(1) $\alpha^\perp \in \mathcal{M}_\omega S$;
(2) $(\alpha^\perp)_\downarrow \in \mathcal{Q}_\omega(\bigcup \alpha)$;
(3) $\alpha = ((\alpha^\perp)_\downarrow)^\perp$.

This justifies the following definition of the antichain representation of $\alpha^{\perp_S}$ in $\mathcal{Q}_\omega S$:

Definition 2.11 Given a set S and an element $\alpha \in \mathcal{Q}_\omega S \cup \mathcal{M}_\omega S$, we define

$$\alpha^\bullet := (\alpha^\perp)_\downarrow.$$

Observe that the above definition does not depend on S: namely, if $S \subseteq S'$ and $\alpha \in \mathcal{Q}_\omega S \subseteq \mathcal{Q}_\omega S'$, then we can index the operation $\perp$ either by S or by S'. However, a straightforward calculation shows that $(\alpha^{\perp_S})_\downarrow = (\alpha^{\perp_{S'}})_\downarrow$. Consequently, the computation of $\alpha^\bullet$ can be executed relative to the least S such that $\alpha \subseteq \mathcal{Q}_\omega S$, which clearly is $\bigcup \alpha$. Together with Proposition 2.10, this also implies that $(\alpha^\bullet)^\bullet = (((\alpha^\perp)_\downarrow)^\perp)_\downarrow = \alpha_\downarrow$.

This paper will see a few key applications for the following *exchange principle*. For its proof, notice that the principle is almost a restatement of Proposition 2.9(1).

Proposition 2.12 (Exchange Principle) *Given a set S, the following are equivalent for any $\alpha \in \mathcal{Q}S$ and $P \in \mathcal{P}(S)$:*
(a) $\exists A \in \alpha \forall a \in A.\, a \in P$;
(b) $\forall B \in \alpha^\perp \exists b \in B.\, b \in P$.
Similarly, for any $\alpha \in \mathcal{Q}_\omega S$ we have the following equivalence:
(a′) $\exists A \in \alpha \forall a \in A.\, a \in P$;
(b′) $\forall B \in \alpha^\bullet \exists b \in B.\, b \in P$.

Remark 2.13 It is well known that $\mathcal{M}$ corresponds to the functor of taking free completely distributive lattice, see for example [24]. In a similar manner, $\mathcal{M}_\omega$ is the free distributive lattice functor [26]. To see this, it is enough to put the set $\alpha \in \mathcal{P}_\omega S$ in correspondence with the term $t_\alpha := \bigwedge_{A \in \alpha_\downarrow} \bigvee A$. Behind the operations $(\,\cdot\,)^\bullet$ and $(\,\cdot\,)^\perp$ we may recognize the action of *dualizing* the term t_α (that is, exchanging meets and joins), followed by rewriting the result, which is now in conjunctive normal form, back into disjunctive normal form.

2.4 Relation Lifting

In the introduction we already defined the operations $\overrightarrow{\mathcal{P}}$, $\overleftarrow{\mathcal{P}}$, $\overline{\mathcal{P}}$, and $\widetilde{\mathcal{M}}$. As mentioned, the notion of a bisimulation between neighborhood models can be nicely expressed using these definitions:

Proposition 2.14 *Let $\mathbb{S}$ and $\mathbb{S}'$ be two models, and let $Z \subseteq S \times S'$. Then Z is a* P*-bisimulation iff $V(s) \cap \mathsf{P} = V(s') \cap \mathsf{P}$ and $(\sigma(s), \sigma'(s')) \in \widetilde{\mathcal{M}}(Z)$, for all $(s, s') \in Z$.*

The following preliminary observations, which can be proved via a straightforward verification, will be used throughout the paper. We shall use ';' and '$\smile$' to denote relational composition and converse, respectively; Δ denotes the identity/diagonal relation, and we write $Gr(f)$ to denote the graph of a function f.

Proposition 2.15 *The operation $\widetilde{\mathcal{M}}$ has the following properties:*

(1) $\widetilde{\mathcal{M}}$ is monotone: if $R \subseteq R'$ then $\widetilde{\mathcal{M}}(R) \subseteq \widetilde{\mathcal{M}}(R')$,

(2) $\widetilde{\mathcal{M}}$ commutes with converse: $\widetilde{\mathcal{M}}(R^\smile) = (\widetilde{\mathcal{M}}(R))^\smile$,

(3) $\widetilde{\mathcal{M}}$ is lax functorial: $\Delta_{\mathcal{Q}S} \subseteq \widetilde{\mathcal{M}}(\Delta_S)$ and $(\widetilde{\mathcal{M}}R_0);(\widetilde{\mathcal{M}}R_1) \subseteq \widetilde{\mathcal{M}}(R_0; R_1)$,

(4) $\widetilde{\mathcal{M}}$ is well defined for $\mathcal{M}$: $(\alpha, \alpha') \in \widetilde{\mathcal{M}}R$ iff $(\alpha^\uparrow, \alpha') \in \widetilde{\mathcal{M}}R$,

(5) $\widetilde{\mathcal{M}}$ commutes with restrictions: $(\widetilde{\mathcal{M}}R) \cap (\mathcal{Q}Y \times \mathcal{Q}Y') = \widetilde{\mathcal{M}}(R \cap (Y \times Y'))$,

(6) $\widetilde{\mathcal{M}}$ is a lax extension of $\mathcal{M}$: $Gr(\mathcal{Q}f) \subseteq \widetilde{\mathcal{M}}(Gr(f))$,

(7) $\widetilde{\mathcal{M}}$ distributes over composition to the left with function graphs:
$\widetilde{\mathcal{M}}(Gr(f);R) = Gr(\mathcal{Q}f);\widetilde{\mathcal{M}}(R)$.

Remark 2.16 It is shown in [32] that, given the properties of the functors $\mathcal{P}$ and $\mathcal{M}$ as *monads* on the category Set, the associated pairs of *directed* relation liftings, $\overrightarrow{\mathcal{P}}/\overleftarrow{\mathcal{P}}$ and $\overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}/\overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}$ respectively, arise in a canonical way. It is interesting to note that, from the logical point of view, the *intersections* $\overline{\mathcal{P}}$ and $\widetilde{\mathcal{M}}$ of these canonically obtained relation liftings turn out to be relevant. The interested reader is referred to [3], where coalgebraic logics are developed on the basis of such directed notions of relation lifting.

3 A Monotone ∇

In this section we introduce the syntax and semantics of $\mathcal{L}_\nabla$, the ∇-based version of monotone modal logic. We prove that this language has the same expressive power as $\mathcal{L}_\diamond$ by showing the interdefinability of ∇ with the pair of modalities $\{\diamond, \Box\}$.

Definition 3.1 Given a set Prop of proposition letters, the set $\mathcal{L}_\nabla(\mathsf{Prop})$ of ∇*-formulas* over Prop, is given by the following (pseudo-)grammar:

$$a \ ::= \ p \mid \neg p \mid \bot \mid \top \mid a \wedge a \mid a \vee a \mid \nabla\alpha,$$

where $p \in \mathsf{Prop}$ and $\alpha \in \mathcal{Q}_\omega\mathcal{L}_\nabla(\mathsf{Prop})$. Given a formula $a \in \mathcal{L}(\mathsf{Prop})$, we let P_a denote the set of variables occurring in a and $d_\nabla(a)$, the ∇*-depth* of a. The latter notion is defined via a straightforward formula induction, with $d_\nabla(\nabla\alpha) := 1 + \max\{\, d_\nabla(a) \mid a \in \bigcup\alpha \,\}$.

Definition 3.2 Let $\mathbb{S} = \langle S, \sigma, V\rangle$ be a monotone neighborhood model. We define the *truth* or *satisfaction relation* $\Vdash \subseteq S \times \mathcal{L}_\nabla$ by induction on the complexity of formulas, the only nontrivial clause (4) already been given in the introduction.

As an immediate consequence of (4) we see that

$$\mathbb{S}, s \Vdash \nabla\alpha \ \text{ iff } \ (\sigma(s), \alpha) \in \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}(\Vdash) \cap \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}(\Vdash), \tag{5}$$

or in words: $\nabla\alpha$ holds at s if every neighborhood U of s supports some set $A \in \alpha$, and every set $A \in \alpha$ holds throughout some neighborhood $U \in \sigma(s)$. Here we say that U *supports* a set A, notation $U \rhd A$, if every formula in A is true at some point in U, and, conversely, that A *holds throughout* U, notation: $U \Vdash \bigvee A$, if every point in U makes some formula in A true:

$$U \Vdash \bigvee A \ \text{ iff } \ (U, A) \in \overrightarrow{\mathcal{P}}(\Vdash) \qquad\qquad U \rhd A \ \text{ iff } \ (U, A) \in \overleftarrow{\mathcal{P}}(\Vdash).$$

Definition 3.3 Let a and b be ($\mathcal{L}_\diamond$- or $\mathcal{L}_\nabla$-)formulas. We write $a \models b$ if $\mathbb{S}, s \Vdash a$ implies $\mathbb{S}, s \Vdash b$, for all pointed models $(\mathbb{S}, s)$. We say that a and b are *equivalent*, notation: $a \equiv b$, if $a \models b$ and $b \models a$, and that a is *valid*, notation: $\models a$, if $\top \models a$.

The following elementary properties of ∇ will turn out to be handy.

Proposition 3.4 *The following hold for any pointed model* $(\mathbb{S}, s)$*:*
(1) $\mathbb{S}, s \Vdash \nabla\varnothing$ *iff* $\sigma(s) = \varnothing$,
(2) $\mathbb{S}, s \Vdash \nabla\{\varnothing\}$ *iff* $\varnothing \in \sigma(s)$.

The main result of this section states that the two languages, $\mathcal{L}_\diamond$ and $\mathcal{L}_\nabla$, are effectively equi-expressive.

Theorem 3.5 *There are effectively defined translations* $(\cdot)^\diamond : \mathcal{L}_\nabla(\mathsf{Prop}) \to \mathcal{L}_\diamond(\mathsf{Prop})$ *and* $(\cdot)^\nabla : \mathcal{L}_\diamond(\mathsf{Prop}) \to \mathcal{L}_\nabla(\mathsf{Prop})$ *such that* $a \equiv a^\diamond$ *for each formula* $a \in \mathcal{L}_\nabla$*, and* $b \equiv b^\nabla$ *for each formula* $b \in \mathcal{L}_\diamond$*.*

In order to prove this theorem, a direct verification will reveal that the modalities $\langle\exists\forall\rangle$ and $\langle\forall\exists\rangle$ are definable in the language $\mathcal{L}_\nabla$.

Proposition 3.6 *The following equivalences hold, for any formula* a*:*

$$\langle\exists\forall\rangle a \equiv \nabla\{\{a\},\{\top\}\} \vee \nabla\{\varnothing\}, \tag{6}$$

$$\langle\forall\exists\rangle a \equiv \nabla\{\{a,\top\}\} \vee \nabla\varnothing\,. \tag{7}$$

Conversely, the nabla modality can be expressed using the box and diamond of monotone modal logic.

Proposition 3.7 *The following equivalence holds for any collection* α *of formula sets:*

$$\nabla\alpha \equiv \bigwedge_{A\in\alpha} \langle\exists\forall\rangle\bigvee A \wedge \bigwedge_{B\in\alpha^\bullet} \langle\forall\exists\rangle\bigvee B. \tag{8}$$

Proof. Fix a pointed model $(\mathbb{S}, s)$. It is immediate by the definitions that

$$(\sigma(s),\alpha) \in \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}(\Vdash) \text{ iff } \mathbb{S}, s \Vdash \bigwedge_{A\in\alpha} \langle\exists\forall\rangle \bigvee A. \tag{9}$$

Also observe that

$$\begin{aligned}
&(\sigma(s),\alpha) \in \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}(\Vdash) \\
&\qquad \text{iff } \forall U \in \sigma(s)\exists A \in \alpha\forall a \in A\exists u \in U \text{ with } u \Vdash a && \text{(definition)}\\
&\qquad \text{iff } \forall U \in \sigma(s)\forall B \in \alpha^\bullet\exists b \in B\exists u \in U \text{ with } u \Vdash b && \text{(exchange principle)}\\
&\qquad \text{iff } \forall B \in \alpha^\bullet\forall U \in \sigma(s)\exists u \in U\exists b \in B \text{ with } u \Vdash b && \text{(quantifier swaps)}\\
&\qquad \text{iff } \mathbb{S}, s \Vdash \bigwedge_{B\in\alpha^\bullet} \langle\forall\exists\rangle\bigvee B\,. && \text{(definitions)}
\end{aligned}$$

By (5), the combination of these observations yields the desired equivalence (8). □

Finally, the proof of Theorem 3.5 follows an obvious induction on formulas, based on the Propositions 3.6 and 3.7.

4 Some algebraic laws

In this section we will see how ∇ interacts with, respectively, the consequence relation $\models$, and the Boolean connectives: $\wedge, \vee$, and $\neg$. (The proofs of this section are deferred to the appendix.)

First we show that the monotone nabla modality is monotone indeed.

Proposition 4.1 *For any $\alpha, \alpha' \in \mathcal{QL}_\nabla$ we have*

$$(\alpha, \alpha') \in \widetilde{\mathcal{M}}(\models) \text{ implies } \nabla\alpha \models \nabla\alpha'. \tag{10}$$

Proof. Assume that $(\alpha, \alpha') \in \widetilde{\mathcal{M}}(\models)$ and let $\mathbb{S}, s$ be a pointed model such that $\mathbb{S}, s \Vdash \nabla\alpha$. Then by definition of $\Vdash$, we have $(\sigma(s), \alpha) \in \widetilde{\mathcal{M}}(\Vdash)$, and so by Proposition 2.15(3) we find that $(\sigma(s), \alpha') \in \widetilde{\mathcal{M}}(\Vdash); \widetilde{\mathcal{M}}(\models) \subseteq \widetilde{\mathcal{M}}(\Vdash; \models)$. Then by $\Vdash; \models \subseteq \Vdash$ and Proposition 2.15(1) we obtain $(\sigma(s), \alpha') \in \widetilde{\mathcal{M}}(\Vdash)$ which shows that $\mathbb{S}, s \Vdash \nabla\alpha'$, as required. $\square$

Next we prove the following distributive law for conjunction.

Definition 4.2 Given a set $F \subseteq \mathcal{L}_\nabla$ of formulas, we let $\mathsf{Conj}(F)$ denote the set of (finite) conjunctions of formulas in F.

Proposition 4.3 *For any $\alpha, \alpha' \in \mathcal{Q}_\omega\mathcal{L}_\nabla$, we have*

$$\nabla\alpha \wedge \nabla\alpha' \equiv \bigvee\{\, \nabla\beta \mid \beta \in \mathcal{Q}(\mathsf{Conj}(\textstyle\bigcup\alpha \cup \bigcup\alpha')) \text{ with } (\beta, \alpha), (\beta, \alpha') \in \widetilde{\mathcal{M}}(\models) \,\}. \tag{11}$$

Proof. It follows by Proposition 4.1 that the left hand side of (11) is a semantic consequence of the right hand side. For the converse, assume that $\mathbb{S}, s \Vdash \nabla\alpha \wedge \nabla\alpha'$. It suffices to come up with a $\beta \in \mathcal{Q}(\mathsf{Conj}(\bigcup\alpha \cup \bigcup\alpha'))$ such that $\mathbb{S}, s \Vdash \nabla\beta$ and $(\beta, \alpha), (\beta, \alpha') \in \widetilde{\mathcal{M}}(\models)$. To this aim, define, for $t \in S$, and $U \in \sigma(s)$,

$$\begin{aligned} A_t &:= \{\, a \in \textstyle\bigcup\alpha \mid \mathbb{S}, t \Vdash a \,\}, & A'_t &:= \{\, a' \in \textstyle\bigcup\alpha' \mid \mathbb{S}, t \Vdash a' \,\}, \\ b_t &:= \textstyle\bigwedge A_t \wedge \bigwedge A'_t, \\ B_U &:= \{\, b_t \mid t \in U \,\}, \\ \beta &:= \{\, B_U \mid U \in \sigma(s) \,\}. \end{aligned}$$

We verify that β has the desired properties. First we check that

$$\mathbb{S}, s \Vdash \nabla\beta. \tag{12}$$

To prove this, think of b as a map $b : S \to \mathcal{L}_\nabla$. It is easy to see that $Gr(b) \subseteq \Vdash$, and that $\beta = (\mathcal{Q}b)(\sigma(s))$. But then it follows from the properties of relation lifting, Proposition 2.15(6), that $(\sigma(s), \beta) \in Gr(\mathcal{Q}b) \subseteq \widetilde{\mathcal{M}}(\Vdash)$. This proves (12).

Second, we prove that

$$(\beta, \alpha) \in \widetilde{\mathcal{M}}(\models) \text{ and } (\beta, \alpha') \in \widetilde{\mathcal{M}}(\models). \tag{13}$$

We show that $(\beta, \alpha) \in \widetilde{\mathcal{M}}(\models)$. If $B \in \beta$, then there is a $U \in \sigma(s)$ such that $B = B_U$. Since $\mathbb{S}, s \Vdash \nabla\alpha$, take an $A \in \alpha$ such that $U \rhd A$. For such an A, if $a \in A$, then $a \in A_u$ for some $u \in U$, and hence the formula $b_u \in B_U = B$ satisfies $b_u \models a$. Conversely, for $A \in \alpha$, let $U \in \sigma(s)$ be such that $U \Vdash \bigvee A$, and consider the set $B_U \in \beta$. For each $b \in B_U$ there is a $u \in U$ with $b = b_u$. From $U \Vdash \bigvee A$, we see that there is an $a \in A$ with $u \Vdash a$. Then $b = b_u \models a$. This proves (13), and hence finishes the proof of the Proposition. □

As a corollary of Proposition 4.3, we may almost eliminate conjunctions from the language $\mathcal{L}_\nabla$, restricting their occurrence to special ones of the form $\bigwedge \Pi \wedge \nabla\alpha$, where Π is a set of literals. This issue will play an important role in the next section.

We now study the behavior of disjunctions occurring directly under the ∇-modality. Observe that an arbitrary such formula can be represented as $\nabla(\,\alpha \cup \{\, C \cup \{\, \bigvee B \,\}\,\}\,)$. First we consider nonempty disjunctions, that is, the case where $B \neq \varnothing$. The following propositions shows how such disjunctions can be *eliminated*.

Proposition 4.4 *For any $\alpha \in \mathcal{Q}_\omega \mathcal{L}_\nabla$ and $B, C \in \mathcal{P}_\omega \mathcal{L}_\nabla$ such that $B \neq \varnothing$, we have*

$$\nabla(\,\alpha \cup \{\, C \cup \{\, \bigvee B \,\}\,\}\,) \equiv \nabla(\,\alpha \cup \{\, C \cup B \,\} \cup \{\, C \cup \{\, b, \top \,\} \mid b \in B \,\}\,)\,. \tag{14}$$

Proof. Fix a pointed model $\mathbb{S}, s$ and abbreviate

$$\begin{aligned} \gamma &= \alpha \cup \{\, C \cup \{\, \bigvee B \,\}\,\}, \\ \delta &= \alpha \cup \{\, C \cup B \,\} \cup \{\, C \cup \{\, b, \top \,\} \mid b \in B \,\}\,. \end{aligned}$$

Recall that

$$\begin{aligned} \mathbb{S}, s \Vdash \nabla\gamma \;\text{ iff } \quad & \forall U \in \sigma(s).\, \Big(\exists A \in \alpha.\, U \rhd A \text{ or } U \rhd C \cup \{\, \bigvee B \,\}\Big) \\ & \text{and } \forall A \in \alpha\, \exists U \in \sigma(s).\, U \Vdash \bigvee A \\ & \text{and } \exists U \in \sigma(s).\, U \Vdash \bigvee C \vee \bigvee B, \end{aligned}$$

while

$$\begin{aligned} \mathbb{S}, s \Vdash \nabla\delta \;\text{ iff } \quad & \forall U \in \sigma(s).\, \Big(\exists A \in \alpha.\, U \rhd A \\ & \qquad\qquad \text{or } U \rhd C \cup B \text{ or } \exists b \in B.\, U \rhd C \cup \{\, b, \top \,\}\Big) \\ & \text{and } \forall A \in \alpha\, \exists U \in \sigma(s).\, U \Vdash \bigvee A \\ & \text{and } \exists U \in \sigma(s).\, U \Vdash \bigvee (C \cup B) \\ & \text{and } \forall b \in B\, \exists U \in \sigma(s).\, U \Vdash \bigvee C \vee b \vee \top. \end{aligned}$$

In order to prove that $\nabla\gamma \equiv \nabla\delta$ we first argue that

$$\begin{aligned} U \rhd C \cup \{\, \bigvee B \,\} \;&\text{ iff }\; \exists b \in B \text{ such that } U \rhd C \cup \{\, b \,\} \\ &\text{ iff }\; U \rhd C \cup B \text{ or } \exists b \in B \text{ such that } U \rhd C \cup \{\, b, \top \,\}\,, \end{aligned}$$

where the second equivalence holds because B is nonempty. Then we note that $\bigvee C \vee \bigvee B \equiv \bigvee(C \cup B)$, and that for a formula b and a neighborhood U of s we *always* have $U \Vdash \bigvee C \vee b \vee \top$. From these obervations, the desired equivalence (14) is immediate. □

The next proposition states how we may eliminate the empty disjunction $\bigvee \varnothing = \bot$, if it occurs directly under a ∇.

Proposition 4.5 *Let $\alpha \in \mathcal{Q}_\omega \mathcal{L}_\nabla$ and $\beta \in \mathcal{P}_\omega \mathcal{L}_\nabla$.*
(1) If $\alpha = \varnothing$ then we have

$$\nabla(\alpha \cup \{ B \cup \{ \bot \} \}) \equiv \bot. \tag{15}$$

(2) If $\alpha \neq \varnothing$, then for any $A \in \alpha$ and $Z \subseteq A \times B$ with $(A, B) \in \overrightarrow{\mathcal{P}}(Z)$, we have

$$\nabla(\alpha \cup \{ B \cup \{ \bot \} \}) \equiv \nabla(\alpha \cup \{ B \cup \{ a \wedge b \mid (a, b) \in Z \} \}) . \tag{16}$$

(3) If the left hand side of (16) *is satisfiable, then there are at least one $A \in \alpha$ and $Z \subseteq A \times B$ with $(A, B) \in \overrightarrow{\mathcal{P}}(Z)$, and the formula $a \wedge b$ satisfiable for each $(a, b) \in Z$.*

Proof. Abbreviate

$$\begin{aligned} \gamma &= \alpha \cup \{ B \cup \{ \bot \} \} , \\ \delta &= \alpha \cup \{ B \cup \{ a \wedge b \mid (a, b) \in Z \} \} . \end{aligned}$$

(1) If $\alpha = \varnothing$ then $\mathbb{S}, s \Vdash \nabla\gamma$ implies the existence of some $U \in \sigma(s)$ such that $U \Vdash \bigvee B \vee \bot$; in particular, we find that $\sigma(s) \neq \varnothing$. However, from $(\sigma(s), \gamma) \in \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}(\Vdash)$ we obtain that $U \rhd B \cup \{ \bot \}$ for each $U \in \sigma(s)$. Clearly this is impossible, which shows that $\nabla\gamma$ is not satisfiable, and hence, equivalent to $\bot$.

(2) We argue first that $(\gamma, \delta) \in \widetilde{\mathcal{M}}(\models)$. Take an arbitrary $C \in \gamma$, then we need to find a $D \in \delta$ such that $(C, D) \in \overleftarrow{\mathcal{P}}(\models)$. This is easy: if $C \in \alpha$, then we take $D := C$, and if $C = B \cup \{ \bot \}$, then we choose $D := B \cup \{ a \wedge b \mid (a, b) \in Z \}$. Conversely, given $D \in \delta$, we need to come up with a $C \in \gamma$ such that $(C, D) \in \overrightarrow{\mathcal{P}}(\models)$. Again, if $D \in \alpha$ we simply take $C := D$. Consider next the case that $D = B \cup \{ a \wedge b \mid (a, b) \in Z \}$, and distinguish cases: if $B \neq \varnothing$, then we may take $C := B \cup \{ \bot \}$, and if $B = \varnothing$ then by totality of Z we also have $A = \varnothing$; in this case we take $C := A$.

We argue next that $(\delta, \gamma) \in \widetilde{\mathcal{M}}(\models)$. First take some $D \in \delta$. If $D \in \alpha$ then define $C := D$, and if $D = B \cup \{ a \wedge b \mid (a, b) \in Z \}$ then put $C := A$. In both cases we see that $(D, C) \in \overleftarrow{\mathcal{P}}(\models)$. Conversely, consider an arbitrary $C \in \gamma$. Again, if $C \in \alpha$ define $D := C$, and if $C = B \cup \{ \bot \}$ take $D := B \cup \{ a \wedge b \mid (a, b) \in Z \}$. In either case it is easily verified that $(D, C) \in \overrightarrow{\mathcal{P}}(\models)$.

(3) Finally, suppose that $\mathbb{S}, s \Vdash \nabla\gamma$. Then there is a $U \in \sigma(s)$ such that $U \Vdash \bigvee B$, and for this U there is an $A \in \gamma$ such that $U \rhd A$. Clearly then $\bot \notin A$, which means that $A \neq B \cup \{ \bot \}$ and so A belongs to α. Define

$$Z := \{ (a, b) \in A \times B \mid U \rhd a \wedge b \}.$$

It is straightforward to verify that this A and this Z have the desired properties: It is obvious that for every $(a,b) \in Z$ the formula $a \wedge b$ is satisfiable. To see that $(A,B) \in \overrightarrow{\mathcal{P}}Z$, take an arbitrary formula $a \in A$. Since $U \rhd A$, there is a $u \in U$ with $\mathbb{S}, u \Vdash a$. Also, from $U \Vdash \bigvee B$, there is a $b \in B$ with $\mathbb{S}, u \Vdash b$. Clearly then $(a,b) \in Z$. $\square$

Remark 4.6 To see why this covers all cases of formulas $\nabla\gamma$ with $\bot \in \bigcup\gamma$, first observe that any such set γ is of the form $\alpha \cup \{ B \cup \{ \bot \} \}$. If the formula $\nabla(\alpha \cup \{ B \cup \{ \bot \} \})$ is *not* satisfiable, then it is equivalent to $\bot$. On the other hand, if it *is* satisfiable, then by Proposition 4.5(3) we may rewrite it to the RHS of (16) for some $A \in \alpha$ and $Z \subseteq A \times B$, with none of the new conjunctions being equivalent to $\bot$. Observe that in the latter case, if B is empty then A and Z must be empty as well. The equation (16) then becomes

$$\nabla(\alpha \cup \{ \{ \bot \} \}) \equiv \nabla(\alpha \cup \{ \emptyset \}) .$$

Finally, because of space limitations, we postpone a discussion of the interaction of ∇ with the negation operator, to an extended version of the paper.

5 Bisimulation Quantifiers and Uniform Interpolation

In this section we will show that, as announced in the Introduction, Monotone Modal Logic enjoys *uniform* interpolation, and we will prove this result, Theorem 1, by showing that the so-called *bisimulation quantifiers* $\exists p$ are effectively expressible in $\mathcal{L}_\nabla$ (and hence in $\mathcal{L}_\Diamond$).

Definition 5.1 Given a modal language $\mathcal{L}(\mathsf{Prop})$ and a notion of bisimulation between pointed models of the language, we obtain the extension $\mathcal{L}^\exists(\mathsf{Prop})$ by adding, for each proposition letter $p \in \mathsf{Prop}$, the *bisimulation quantifier* $\exists p$ to the language. The semantics of this quantifier is given by (2).

When we say that the bisimulation quantifiers are effectively expressible in $\mathcal{L}$, we mean that there is an effective translation mapping formulas in $\mathcal{L}^\exists$ to equivalent formulas in $\mathcal{L}$. In our case, we will define such a translation in two steps. First we use the results of the previous section to rewrite an $\mathcal{L}_\nabla$-formula a into its so-called *transparent* normal form a^n. Then we show that transparent formulas admit a simple inductive definition of the translation.

Definition 5.2 The fragment $\mathcal{L}_\nabla^-$ of *disjunctive* formulas in $\mathcal{L}_\nabla$ is given by the following grammar:

$$a \ ::= \ \top \mid \bot \mid \bigwedge\Pi \mid \bigwedge\Pi \wedge \nabla\alpha \mid a \vee a,$$

where Π is a set of *literals*, and $\alpha \in \mathcal{Q}_\omega\mathcal{L}_\nabla^-$. A formula $a \in \mathcal{L}_\nabla^-$ is called *transparent* if in every subformula $\nabla\alpha$ of a, every formula in $\bigcup\alpha$ is satisfiable.

Proposition 5.3 *There is an effective algorithm rewriting any formula $a \in \mathcal{L}_\nabla$ into an equivalent transparent formula a^n.*

The *proof* of this proposition proceeds via a straightforward induction on the ∇-depth of a, on the basis of the Propositions 4.3 and 4.5. We skip the details, and move on to our inductive definition of the bisimulation quantifiers.

Definition 5.4 Given $p \in \mathsf{Prop}$, we inductively define the map $\tau_p : \mathcal{L}^-_\nabla(\mathsf{Prop}) \to \mathcal{L}^-_\nabla(\mathsf{Prop})$ by:

$$\begin{array}{lcl}
\tau_p(\top) & := & \top \\
\tau_p(\bot) & := & \bot \\
\tau_p\left(\bigwedge\Pi\right) & := & \begin{cases} \bot & \text{if } \{\, p, \neg p \,\} \subseteq \Pi \\ \bigwedge(\Pi \setminus \{\, p, \neg p \,\}) & \text{otherwise} \end{cases} \\
\tau_p\left(\bigwedge\Pi \wedge \nabla\alpha\right) & := & \tau_p\left(\bigwedge\Pi\right) \wedge \nabla(\mathcal{Q}\tau_p)(\alpha) \\
\tau_p(a \vee b) & := & \tau_p(a) \vee \tau_p(b)\,.
\end{array}$$

The following proposition shows that for transparent formulas, the above definition satisfies the required properties.

Proposition 5.5 *Let $p \in \mathsf{Prop}$ be some proposition letter. For any disjunctive formula $a \in \mathcal{L}^-_\nabla(\mathsf{Prop})$ we have $\tau_p(a) \in \mathcal{L}^-_\nabla(\mathsf{P}_a \setminus \{\, p \,\})$. Moreover, if a is transparent, then $\tau_p(a) \equiv \exists p.a$.*

Proof. The first statement of the Proposition is a straightforward consequence of the definitions. The second statement is proved by induction on the definition of transparent disjunctive formulas; the inductive clauses are immediate consequences of Proposition 5.6 below. $\square$

The following Proposition is the key technical lemma of this paper. In particular, we show that the bisimulation quantifiers distributes over ∇, provided that all formulas under the nabla are *satisfiable*. This proviso explains why in Proposition 5.5 we can only prove that $\tau_p(a) \equiv \exists p.a$ for transparent a.

Proposition 5.6 *The bisimulation quantifier $\exists p$ satisfies the following properties:*

(B1) $\exists p.(a \vee b) \;\equiv\; \exists p.a \vee \exists p.b$*;*

(B2) $\exists p.\nabla\alpha \equiv \nabla(\mathcal{Q}\exists p)(\alpha)$*, provided every* $a \in \bigcup\alpha$ *is satisfiable;*

(B3) $\exists p.\left(\bigwedge\Pi \wedge \nabla\alpha\right) \equiv \begin{cases} \bot & \text{if } \{\, p, \neg p \,\} \subseteq \Pi, \\ \bigwedge(\Pi \setminus \{\, p, \neg p \,\}) \wedge \exists p.\nabla\alpha & \text{otherwise.} \end{cases}$

In the formulation of condition (B2) above, it is convenient to see the quantifier $\exists p$ as a function on the language $\mathcal{L}$ (mapping a formula a to the formula $\exists p.a$), so that we may apply the functor $\mathcal{Q}$ to it and obtain a map $\mathcal{Q}\exists p : \mathcal{QL} \to \mathcal{QL}$.

Proof. Since the items (B1) and (B3) follow by a routine argument, we focus on the proof of (B2). Fix a pointed model $\mathbb{S}, s_0$. We will show that

$$\mathbb{S}, s_0 \Vdash \exists p.\nabla\alpha \text{ iff } \mathbb{S}, s_0 \Vdash \nabla(\mathcal{Q}\exists p)(\alpha). \tag{17}$$

For the direction from left to right of (17), assume that $\mathbb{S}, s_0 \Vdash \exists p.\nabla\alpha$. Then there is a pointed model $\mathbb{S}', s_0'$ and an up-to-p bisimulation Z such that $Z : \mathbb{S}, s_0 \simeq_p \mathbb{S}', s_0'$ and $\mathbb{S}', s_0' \Vdash \nabla\alpha$. It follows that $(\sigma(s_0), \sigma'(s_0')) \in \widetilde{\mathcal{M}}(Z)$ and $(\sigma'(s_0'), \alpha) \in \widetilde{\mathcal{M}}(\Vdash)$, and so by Proposition 2.15(3,1) we find $(\sigma(s_0), \alpha) \in \widetilde{\mathcal{M}}(Z; \Vdash)$. However, since Z is an up-to-p bisimulation, it follows that $Z; \Vdash \subseteq \Vdash; Gr(\exists p)^{\smile}$. From this we may infer that $(\alpha, \sigma(s_0)) \in \left(\widetilde{\mathcal{M}}(Z; \Vdash)\right)^{\smile} = \left(\widetilde{\mathcal{M}}(\Vdash; Gr(\exists p)^{\smile})\right)^{\smile} = \widetilde{\mathcal{M}}\left(Gr(\exists p); \Vdash^{\smile}\right) = Gr(\mathcal{Q}\exists p); \widetilde{\mathcal{M}}\left(\Vdash^{\smile}\right) = Gr(\mathcal{Q}\exists p); \left(\widetilde{\mathcal{M}}(\Vdash)\right)^{\smile}$. But $(\alpha, \sigma(s_0)) \in (Gr(\mathcal{Q}\exists p)); \left(\widetilde{\mathcal{M}}(\Vdash)\right)^{\smile}$ is another way of saying that $((\mathcal{Q}\exists p)(\alpha), \sigma(s_0)) \in \left(\widetilde{\mathcal{M}}(\Vdash)\right)^{\smile}$, or equivalently, $(\sigma(s_0), (\mathcal{Q}\exists p)(\alpha)) \in \widetilde{\mathcal{M}}(\Vdash)$. This means that $\mathbb{S}, s_0 \Vdash \nabla(\mathcal{Q}\exists p)(\alpha)$, as required.

For the converse direction of (17), assume that $\mathbb{S}, s_0 \Vdash \nabla(\mathcal{Q}\exists p)(\alpha)$. In order to prove that $\mathbb{S}, s_0 \Vdash \exists p.\nabla\alpha$, we need to construct some pointed model $(\mathbb{S}', s_0')$ such that $\mathbb{S}, s_0 \simeq_p \mathbb{S}', s_0'$ and $\mathbb{S}', s_0' \Vdash \nabla\alpha$. For this purpose, consider the set

$$P := \{ (s, a) \in \bigcup\sigma(s_0) \times (\bigcup\alpha \cup \{ \top \}) \mid \mathbb{S}, s \Vdash \exists p.a \}.$$

For $(s, a) \in P$, pick a pointed model $(\mathbb{T}_{s,a}, t_{s,a})$ with $\mathbb{S}, s \simeq_p \mathbb{T}_{s,a}, t_{s,a}$ and $\mathbb{T}_{s,a}, t_{s,a} \Vdash a$. (In the case that $a = \top \notin \bigcup\alpha$, since $\exists p.\top$ is equivalent to $\top$, we may chose $\mathbb{T}_{s,a} := \mathbb{S}$ and $t_{s,a} := s$.) Also, for $a \in \bigcup\alpha$, let (T_a, t_a) be an arbitrary pointed model of a — thus $T_a, t_a \Vdash a$. Note that here we use the fact that all formulas in $\bigcup\alpha$ are satisfiable.

We define the model $\mathbb{S}'$ as follows. Its domain is given as the disjoint union

$$S' := \{ s_0' \} \uplus \biguplus\{ T_{s,a} \mid (s, a) \in P \} \uplus \biguplus\{ T_a \mid a \in \bigcup\alpha \}.$$

The neighborhood map $\sigma' : S' \to \mathcal{M}S'$ can be described as follows. For $s' \neq s_0'$, we put

$$\sigma'(s') := (\sigma_x(s'))^{\uparrow},$$

where $x \in P \cup \bigcup\alpha$ is the unique x such that $s' \in T_x$, σ_x is the neighborhood map of $\mathbb{T}_x$, and $(\cdot)^{\uparrow}$ denotes the operation of closing under supersets of S', cf. Definition 2.6. For the definition of $\sigma'(s_0')$, we first define, for $U \in \sigma(s_0)$, the set

$$U' := \{ t_{s,a} \mid (s, a) \in P, s \in U \}.$$

Second, by $\mathbb{S}, s \Vdash \nabla(\mathcal{Q}\exists p)(\alpha)$, we may pick, for each $A \in \alpha$, some $U_A \in \sigma(s_0)$ such that $U_A \Vdash \bigvee\{ \exists p.a \mid a \in A \}$. Define

$$U_A^{\bullet} := \{ t_{s,a} \mid (s, a) \in P,\ s \in U_A, a \in A \} \cup \{ t_a \mid a \in A \}$$

and

$$\sigma'(s_0') := (\{ U' \mid U \in \sigma(s_0) \} \cup \{ U_A^{\bullet} \mid A \in \alpha \})^{\uparrow}.$$

With these definitions, each $\sigma(s')$ is indeed an upward closed collection of subsets of S'. We complete the construction of the model $\mathbb{S}'$ by defining the following coloring $V' : S' \to \mathcal{P}(\mathsf{Prop})$:

$$V'(s') := \begin{cases} V_x(s') & \text{if } s' \in T_x, \\ V(s_0) & \text{if } s' = s'_0. \end{cases}$$

In the sequel, we will use without proof the fact that for all points $s' \in S' \setminus \{ s'_0 \}$, there is a unique index x such that $s \in T_x$, and that $\mathbb{S}', s' \simeq \mathbb{T}_x, s'$. From this it follows that $\mathbb{S}', t_{s,a} \Vdash a$ for each $(s,a) \in P$, and that $\mathbb{S}', t_a \Vdash a$ for each $a \in \bigcup \alpha$.

In order to show that $\mathbb{S}, s_0 \Vdash \exists p.\nabla\alpha$, it suffices to prove (18) and (20) below. First we claim that

$$\mathbb{S}, s_0 \simeq_p \mathbb{S}', s'_0. \tag{18}$$

To prove this, take some up-to-p bisimulation $Z_{s,a} : \mathbb{S}, s \simeq_p \mathbb{T}_{s,a}, t_{s,a}$ for each $(s,a) \in P$, and let

$$Z := \{ (s_0, s'_0) \} \cup \bigcup \{ Z_{s,a} \mid (s,a) \in P \}.$$

We claim that $Z \subseteq S \times S'$ is an up-to-p bisimulation between $\mathbb{S}$ and $\mathbb{S}'$.

By Proposition 2.14, it suffices to verify, for each pair $(s,s') \in Z$, that the colorings of s and s' agree on all proposition letters other than p, and that $(\sigma(s), \sigma'(s')) \in \widetilde{\mathcal{M}}(Z)$. These facts require only a routine check for the pairs $(s,s') \neq (s_0, s'_0)$, and so we focus on the pair $(s_0, s'_0) \in Z$. Since $V'(s'_0) = V(s_0)$ by definition, it remains to show that

$$(\sigma(s_0), \sigma'(s'_0)) \in \widetilde{\mathcal{M}}(Z). \tag{19}$$

For a proof of (19), first take an arbitrary neighborhood U of s_0. To see that $(U, U') \in \overleftarrow{\mathcal{P}}(Z)$, consider some $u' \in U'$. By definition of U' we must have $u' = t_{s,a}$ for some $(s,a) \in P$ with $s \in U$. From this it is immediate that $(s, u') \in Z_{s,a} \subseteq Z$, as required. This proves that $(\sigma(s_0), \sigma'(s'_0)) \in \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}(Z)$. Conversely, consider a set $W \in \sigma'(s'_0)$ and distinguish cases. If $W = U'$ for some $U \in \sigma(s_0)$, then for any $u \in U$ we have $t_{u,\top} \in U'$, and $(u, t_{u,\top}) \in Z_{u,\top} \subseteq Z$, whence $(U, W) \in \overrightarrow{\mathcal{P}}(Z)$. Otherwise, we have $W = U^\bullet_A$ for some $A \in \alpha$, and we claim that $(U_A, W) \in \overrightarrow{\mathcal{P}}(Z)$. To see this, recall that by definition, for any $u \in U_A$ there is some $a \in A$ such that $\mathbb{S}, u \Vdash a$. From this it follows that $t_{u,a} \in U^\bullet_A$ and since $(u, t_{u,a}) \in Z_{u,a} \subseteq Z$, we find that $(U_A, W) \in \overrightarrow{\mathcal{P}}(Z)$ indeed. This means that $(\sigma(s_0), \sigma'(s'_0)) \in \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}(Z)$, which finishes the proof of (19). Thus we have proved that $Z : \mathbb{S}, s_0 \simeq_p \mathbb{S}', s'_0$, which establishes (18).

Our second claim is that

$$\mathbb{S}', s'_0 \Vdash \nabla\alpha. \tag{20}$$

To see this, first consider an arbitrary set $A \in \alpha$. Any point $s' \in U^\bullet_A$ is either of the form $t_{s,a}$ for a pair $(s,a) \in P$ with $s \in U_A$ and $a \in A$, or of the form t_a for some $a \in A$. In both cases, we see that there exists some $a \in A$ such that $\mathbb{S}', s' \Vdash a$. This suffices to show that $(U^\bullet, A) \in \overrightarrow{\mathcal{P}}(\Vdash)$, and since $U^\bullet \in \sigma'(s'_0)$, it follows that $(\sigma'(s_0), \alpha) \in \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}(\Vdash)$.

Conversely, take an arbitrary $W \in \sigma(s'_0)$, so that W is either of the form U' for some $U \in \sigma(s_0)$, or of the form $U^\bullet_A$ for some $A \in \alpha$. In the first case, there is some $A' \in \alpha$ such that $U \rhd_{\mathbb{S}} \{ \exists p.a \mid a \in A' \}$. It follows that, for each $a \in A'$, there is a $u \in U$ such that $\mathbb{S}, u \Vdash a$, and so the point $t_{u,a}$ belongs to U' and $\mathbb{S}', t_{u,a} \Vdash a$. That is, $W \rhd_{\mathbb{S}'} A'$. In the second case, for any $a \in A$ we may take the point $t_a \in U^\bullet_A$, which satisfies $\mathbb{S}', t_a \Vdash a$; thus we see that $W \rhd_{\mathbb{S}'} A$. This means that $(\sigma'(s'_0), \alpha) \in \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}(\Vdash)$, and so we have proved that $(\sigma'(s'_0, \alpha) \in \widetilde{\mathcal{M}}(\Vdash)$, which suffices for proving (20).

This finishes the proof of the direction from right to left of (17). $\square$

Finally, the Uniform Interpolation Theorem 1 is an immediate consequence of Proposition 5.5.

Proof of Theorem 1 Fix a formula $a \in \mathcal{L}_\diamond(\mathsf{Prop})$. Given a set $\mathsf{Q} \subseteq \mathsf{P}_a$, write $\mathsf{P}_a \setminus \mathsf{Q} = \{ p_0, \ldots, p_{n-1} \}$. We define

$$a_{\mathsf{Q}} := \left(\tau_{p_0} \cdots \tau_{p_{n-1}} \left(\left(a^\nabla\right)^n\right)\right)^\diamond .$$

It follows that $a_{\mathsf{Q}} \in \mathcal{L}_\diamond(\mathsf{Q})$, and that

$$a_{\mathsf{Q}} \equiv \exists p_0. \cdots \exists p_{n-1}.a.$$

So in order to verify that a_{Q} is the required uniform interpolant of a with respect to Q, it suffices to check (1) for an arbitrary b with $\mathsf{P}_a \cap \mathsf{P}_b \subseteq \mathsf{Q}$. First assume $a \models b$. In order to prove that $a_{\mathsf{Q}} \models b$, take an arbitrary pointed model $\mathbb{S}, s$ such that $\mathbb{S}, s \Vdash a_{\mathsf{Q}}$. It follows that there are pointed models $(\mathbb{S}_i, s_i)_{0 \le i \le n}$ such that $(\mathbb{S}, s) = (\mathbb{S}_0, s_0)$, $\mathbb{S}_i, s_i \simeq_{p_i} \mathbb{S}_{i+1}, s_{i+1}$ for all i, and $\mathbb{S}_n, s_n \Vdash a$. Then by assumption we have $\mathbb{S}_n, s_n \Vdash b$, and since none of the p_i occurs in b, it follows by bisimulation invariance that $\mathbb{S}_i, s_i \Vdash b$, for all i. In particular, we find that $\mathbb{S}, s \Vdash b$, as required. Conversely, if $a_{\mathsf{Q}} \models b$ then by $a \models a_{\mathsf{Q}}$ we immmediately obtain that $a \models b$. $\square$

6 Conclusions & Questions

In this paper we have introduced, for monotone modal logic, a modality ∇ that intuitively simulates in this context the cover modality for modal logic **K**. We have then defined a modal language based on ∇ and proved that this language is equi-expressive with the standard one. Using some algebraic laws satisfied by ∇ we have shown that each formula is equivalent to a formula which is transparent. Transparent formulas should be thought of as formulas in a rather special, disjunctive normal form. For such formulas it is relatively easy to compute uniform interpolants. Consequently, we arrived at our main result stating that all formulas of monotone modal logic have uniform interpolants in **M**.

On the basis of our results we see various way to continue. First of all, we might improve on the results presented here. For instance, we are curious after the optimal *size* of the uniform interpolant is, and after the computational *complexity* of computing it. Note that the present construction is based on rewriting $\mathcal{L}_\nabla$-formulas into transparent normal form, and this process, involving satisfiability checks of very complex formulas (Proposition 4.5), and exponential blow-ups each time a conjunction is pushed down

(Proposition 4.3), is probably not optimal in terms of efficiency. We hope to come back to this issue in an extended version of this paper.

Another natural direction for future research would be to look for variations and extensions of our uniform interpolation result. To start with, we are very much interested whether, analogous to the results of d'Agostino and Hollenberg on the modal μ-calculus [10], the extension of monotone modal logic with fixpoint operators has uniform interpolation as well. A related direction would be to investigate the existence of uniform interpolants in specific monotone logics, whether they are defined as axiomatic extensions of **M**, or semantically by means of classes of monotone neighborhood frames. Conversely, we would like to know whether our results generalize to classical modal logic **C**, the logic of arbitrary (that is, not necessarily monotone) neighborhood models. However, taking in account that in normal modal logic, uniform interpolation does not transpose easily from one variety to another, one should expect the same sort of phenomenon in the generalized setting.

On a more abstract level, the approach we have followed might be considered naive, as we mimicked, within monotone modal logic the approach towards coalgebraic logic taken by Moss [25], but based on a different notion of relation lifting than the canonical one. However, the fact that we obtained such a powerful result, may indicate there are some general categorical principles underlying our naive approach. This would be in accordance with the fact that for a functor T that does not preserve weak pullbacks, the notion of bisimilarity based on the standard relation lifting $\overline{T}$ is not the appropriate one. Therefore, our work suggests new directions of research in the area of coalgebras and category theory. Ideas on relation lifting from [18,32] might be useful here.

Finally, we believe that ∇-based monotone modal logic is of interest in its own right, and we plan to study of it in more detail. In particular, we conjecture that the ∇-laws of Section 4, augmented with appropriate axioms expressing the interaction between ∇ and the Boolean negation, provide a sound and complete *axiomatization* for the set of valid $\mathcal{L}_\nabla$-formulas, and we hope to report on this in an extended version of this paper.

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