Proofs, Disproofs, and Their Duals

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Abstract

Bi-intuitionistic logic, also known as Heyting-Brouwer logic or subtractive logic, is extended in various ways by a strong negation connective used to express commitments arising from denials. These logics have been introduced and investigated in [48]. In the present paper, an inferentialist semantics in terms of proofs, disproofs, and their duals is developed. Whereas the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic uses just the notion of proof as primitive, and López-Escobar's inferentialist interpretation of Nelson's logics with strong negation utilizes only the notions of proof and disproof as primitive, the inferentialist interpretation of bi-intuitionistic logic with strong negation employs the four notions of proofs, disproofs, dual proofs, and dual disproofs as primitive concepts.

Keywords: proofs, disproofs, dual proofs, dual disproofs, proof-theoretic semantics, constructive logic, connexive logic, constructive negation, constructive implication, constructive co-implication.

1 Introduction

1.1 Inferential status and speech acts

It seems to be an accepted view that assertion and denial are particularly important speech acts in the context of a use-based, inferentialist account of linguistic meaning. In particular, the idea is that the rules of use that determine the meaning of linguistic expressions provide a basis for warranted assertions and denials. In order to make an assertion, it is enough to seriously utter a sentence, for example the sentence 'Mary is beautiful'. In order to obtain an absolutely clear case of denying that Mary is beautiful, the contrary predicate 'is ugly' may be used, i.e., the sentence 'Mary is ugly' may be seriously uttered. Instead of replacing the adjective 'beautiful' by another, contrary item from the lexicon, namely the adjective 'ugly', one may employ a suitable unary negation connective, $\sim$ 'it is definitely false that'. It is definitely false that Mary is beautiful if and only if Mary is ugly. The "only if" may not be clear. If it is definitely false that Mary is beautiful, then Mary not just fails to be beautiful, but she is ugly. It is then denied that Mary is beautiful by seriously uttering the negated sentence '$\sim$ Mary is beautiful'. This move is supported by the existence of more systematically and regularly connected pairs of contrary predicates in the lexicon: 'sane' versus 'insane', 'believes' versus 'disbelieves', 'desirable' versus 'undesirable', etc. The prefixes 'in', 'dis' and 'un' suggest the introduction of the negation connective $\sim$, so that a denial of a sentence A may be represented as an assertion of $\sim A$.

Negation can be iterated. Is a denial of $\sim A$ an assertion of A? Can denying be iterated? Can asserting be iterated? It seems plausible to assume that a speaker may assert that Mary is beautiful not only by seriously uttering the sentence 'Mary is beautiful', but also by seriously uttering the sentence 'I assert that Mary is beautiful'. Similarly, a speaker may deny that Mary is beautiful not just by seriously uttering the sentences 'Mary is ugly', but also by seriously uttering the sentence 'I deny that Mary is beautiful'. Clearly, first- and other-person asserting-that-ascriptions and denying-that-ascriptions *may* be iterated. A sentence such as 'I deny that I deny that Mary is beautiful' is perfectly grammatical, though perhaps difficult to parse. Seriously uttering this sentence amounts to performing the same speech act as uttering the perhaps more idiomatic sentence 'I deny that Mary is ugly'. A clear case of denying that Mary is ugly is seriously uttering the sentence 'Mary is beautiful'. A denial of $\sim A$ thus seems to be an assertion of A, and recall that an assertion of $\sim A$ was introduced as a denial of A.

The notions of assertion and denial stand in a close relation to the notions of proof and disproof, respectively. If I assert that A, then I commit myself to be ready to prove A, and if I deny that A, then I commit myself to be ready to disprove A. Assertion and denial are basic speech acts which are insensitive to the complexity and composition of the asserted or denied sentence A. No matter how complex A may be and no matter how A is composed, in order to assert or deny A, it is enough to seriously utter the sentence A or its strong negation $\sim A$. A (canonical) proof or a (canonical) disproof of a sentence A, however, *is* sensitive to the complexity and composition of A. A canonical proof of a conjunction $(A \wedge B)$, for example, requires a proof of A and a proof of B, whereas a canonical disproof of $(A \vee B)$ requires a disproof of A and a disproof of B.

If we look only at proofs and disproofs of elementary sentences representable by atomic formulas of a propositional or first-order language, then proofs and disproofs often are basic acts. We can take up an example provided by A. Grzegorczyk [13]. Suppose that l is a yellow lemon. We may prove that l is yellow just by drawing visual attention to l, and we may disprove that l is red again just by drawing visual attention to l. The falsification of the proposition that l is red is as direct as the verification of the proposition that l is yellow. Neither would we attempt to disprove that l is red by leading the assumption that l is red to an absurdity, nor would we attempt to prove that l is yellow by leading the assumption that l fails to be yellow to an absurdity. We would, under normal circumstances at least, just point to the colour of l. It might be objected that the provability of an elementary sentence such as 'l is yellow' requires a theory and that in verifying by eye that l is yellow, we do not just see that l is yellow but *infer* that l is yellow from a theory based on our visual experience. But then in falsifying by eye that l is red, we still do not seem to lead the assumption that l is red to an absurdity. If in the verification case we directly *infer* from (a theory based on) our visual experience that l is yellow, then in the falsification case it seems that we directly infer from (a theory based on) our visual experience that l is definitely not red. Therefore, *if* disproving by eye that l is red is conceived of as an inference of the proposition that l is definitely not red, then this "definitely not" is not a so-called negation as inconsistency. In other words 'l is definitely not red' is not to be understood as 'l implies absurdity', cf. [9,42].

What is absurdity? A sentence expresses absurdity, the absurd proposition, if in every model, the sentence fails to be true. If we consider possible worlds models, a sentence expresses absurdity, if the sentence fails to be true at every possible world in every model. Possible worlds are often conceived of as classical models satisfying the principle of bivalence. But they may also be conceived of as information states that may or may not support the truth or the falsity of propositions. If absence of truth is distinguished from falsity, so that the principle of bivalence is violated, a sentence may express absurdity without being false in every model or false at every state in every model. A sentence thus expresses absurdity if it is never true, and, *in general*, a reduction to absurdity is a reduction to non-truth. Of course, we may then also consider reductions to non-falsity. A sentence expresses non-falsity, if it is never false.

If an act of assertion commits a speaker to be ready to prove the asserted proposition, and an act of denial commits a speaker to be ready to disprove the denied proposition, one may wonder what kind of action is such that it commits a speaker to be ready to reduce the assumption that a certain proposition A is true to absurdity (or, more generally, to non-truth) and what kind of action is such it commits a speaker to be ready to reduce the assumption that A is definitely false to non-truth. It seems that the *first* kind of commitment comes with asserting that nothing supports the truth of A. If I assert that nothing supports the truth of the sentence 'Person b stabbed person c', I am committed to be ready to show that any piece of information (in particular any piece of information that seems to establish the truth of the assumption that b stabbed c) fails to establish the truth of the assumption that b stabbed c. If there is a witness who claims to have seen that b stabbed c, for example, I may point out that the witness used to be extremely unreliable on previous occasions. Proceeding in this way, I may try to show that there is no conclusive evidence in favour of 'b stabbed c'.

What makes it difficult, perhaps, to see the difference between disproofs and reductions to absurdity is that one might hold that every direct falsification of A also reduces the assumption that A to absurdity. If I present a group of very reliable witnesses who confirm that b was not at the crime scene, this may be viewed as a direct falsification of 'b stabbed c', in addition leading the assumption that b stabbed c to absurdity. But, firstly, this does not show that there is no difference between disproving and reducing to absurdity, and, secondly, note that information may be contradictory. Someone else might present another group of highly reliable witnesses who claim that they saw that b stabbed c, so that the available testimony both supports the truth and supports the falsity of 'b stabbed c'. Thus, it is not at all clear that disprovability always implies reducibility to non-truth. Indeed, the implication may fail.

The *second* kind of commitment appears to come with asserting that no information supports the falsity of assumption A. If I assert that no information supports the falsity of 'b stabbed c', I am committed to be ready to show that the assumption that b definitely did not stab c leads to absurdity. Again, I might try to point to certain facts that are incompatible with the assumption under consideration, although they do not prove that b stabbed c. I might, for example, point out that b's fingerprints can be found on the dagger that has been removed from c's corpse.

The view that the denial of a sentence s can be profitably analyzed as the assertion of a suitable negation of s is contentious. According to Greg Restall [31]

> [d]enial is not to be analysed as the assertion of a negation,

whereas Bryson Brown [3, p. 646] explains that he has

> a modest proposal: negation is *denial* in the object language.

Graham Priest [27, p. 105] concedes that the uttering of a negated sentence sometimes may be interpreted as a denial but holds that "asserting a negation (in the Fregean sense) is not necessarily a denial." Priest regards rejection as the linguistic expression of denial and takes rejecting something to be putting a bar on accepting it. "When justified, it is so because there is evidence against the claim: positive grounds for keeping it out of one's beliefs" [27, p. 103]. This exclusion from belief is stronger than agnosticism (absence of belief) but, as it seems, weaker than disbelief. Timothy Williamson [51, p. 10] explains that "we can regard assertion as the verbal counterpart of judgement and judegement as the occurrent form of belief". The association of assertions with proofs and denials with disproofs takes the negative judgement of denial as the occurrent form of disbelief and not as the occurrent form of refusal from belief. [1]

In this paper I would like to discuss logics in which it is important to distinguish between provability, disprovability, and their duals. The term 'duality' has several meanings even in mathematics. In one usage the concept of duality is related to order reversal. In this sense, the dual of provability is reducibility to non-truth. The dual of disprovability is reducibility to non-falsity. The picture summarized in Table 1 emerges.

	inferential status	related speech act
$\varnothing \vdash A$	A is provable	to assert that A
$\varnothing \vdash \sim A$	A is disprovable	to deny that A
$A \vdash \varnothing$	A is reducible to non-truth	to assert that no information supports the truth of A
$\sim A \vdash \varnothing$	A is reducible to non-falsity	to assert that no information supports the falsity of A

Table 1
Speech acts and the inferential status of propositions.

[1] I intend to discuss the relation between assertion, denial, and negation in more detail in a separate paper.

1.2 Inferential relations and logical operations

If A is provable, then it is warranted to assert that A, if A is disprovable, then it is warranted to deny that A, if A is reducible to non-truth, then it is warranted to assert that no information supports the truth of A, and if A is reducible to non-falsity, then it is warranted to assert that no information supports the falsity of A.

The above considerations on the inferential status of a sentence A can be generalized to proofs from a finite set of sentences assumed to true $A_1, \ldots, A_n$ and reductions from a finite set of sentences $A_1, \ldots, A_n$ assumed not to be true. If the expression 'assumptions' is reserved for sentences assumed to be true, there seems to be a semantic gap in English and other natural languages, as there is no idiomatic term for sentences assumed not to be true. Let us agree to call sentences assumed not to be true *counterassumptions*. Sentences assumed to be false may be called *rejections* (or repudiations), so that sentences assumed not to be false might be called *counterrejections* (counterrepudiations). Table 2 lists eight different kinds of inferential relations.

	inferential relation
$A_1, \ldots, A_n \vdash A$	A is provable from assumptions $A_1, \ldots, A_n$
$A_1, \ldots, A_n \vdash {\sim}A$	A is disprovable from assumptions $A_1, \ldots, A_n$
$A \vdash A_1, \ldots, A_n$	A is reducible to absurdity from counterassumptions $A_1, \ldots, A_n$
${\sim}A \vdash A_1, \ldots, A_n$	A is reducible to non-falsity from counterassumptions $A_1, \ldots, A_n$
${\sim}A_1, \ldots, {\sim}A_n \vdash A$	A is provable from rejections $A_1, \ldots, A_n$
${\sim}A_1, \ldots, {\sim}A_n \vdash {\sim}A$	A is disprovable from rejections $A_1, \ldots, A_n$
$A \vdash {\sim}A_1, \ldots, {\sim}A_n$	A is reducible to absurdity from counterrejections $A_1, \ldots, A_n$
${\sim}A \vdash {\sim}A_1, \ldots, {\sim}A_n$	A is reducible to non-falsity from counterrejections $A_1, \ldots, A_n$

Table 2
Inferential relations.

If we want to reduce the inferential relation between the sentences $A_1, \ldots, A_n$ and the sentence A to the inferential status of a single formula, we may use suitable connectives: conjunction $\wedge$, disjunction $\vee$, implication $\rightarrow$, and the less well-known co-implication $\prec$, see Table 3.

We thereby arrive at the following vocabulary: $\{\wedge, \vee, \rightarrow, \prec, \sim\}$. Whereas $\wedge, \vee, \rightarrow$, and $\prec$ may be seen to emerge from the reduction of inferential relations to inferential status stated in Table 3, $\sim$ reflects the distinction between provability and disprovability. Conjunction $\wedge$ combines formulas in antecedent position, i.e., on the left of $\vdash$, and disjunction combines formulas in succedent position, i.e., on the right of $\vdash$. Implication

inferential relation	inferential status
$A_1, \ldots, A_n \vdash A$	$\varnothing \vdash (A_1 \wedge \ldots \wedge A_n) \rightarrow A$
$A_1, \ldots, A_n \vdash {\sim}A$	$\varnothing \vdash (A_1 \wedge \ldots \wedge A_n) \rightarrow {\sim}A$
$A \vdash A_1, \ldots, A_n$	$A \prec (A_1 \vee \ldots \vee A_n) \vdash \varnothing$
${\sim}A \vdash A_1, \ldots, A_n$	${\sim}A \prec (A_1 \vee \ldots \vee A_n) \vdash \varnothing$
${\sim}A_1, \ldots, {\sim}A_n \vdash A$	$\varnothing \vdash ({\sim}A_1 \wedge \ldots \wedge {\sim}A_n) \rightarrow A$
${\sim}A_1, \ldots, {\sim}A_n \vdash {\sim}A$	$\varnothing \vdash ({\sim}A_1 \wedge \ldots \wedge {\sim}A_n) \rightarrow {\sim}A$
$A \vdash {\sim}A_1, \ldots, {\sim}A_n$	$A \prec ({\sim}A_1 \vee \ldots \vee {\sim}A_n) \vdash \varnothing$
${\sim}A \vdash {\sim}A_1, \ldots, {\sim}A_n$	${\sim}A \prec ({\sim}A_1 \vee \ldots \vee {\sim}A_n) \vdash \varnothing$

Table 3
From inferential relations to inferential status.

is a vehicle for registering formulas that appear in antecedent position in succedent position, and co-implication is a vehicle for registering formulas that appear in succedent position in antecedent position. We read $A \prec B$ as "B co-implies A" or as "A excludes B". Whereas implication is the residuum of conjunction, co-implication is the residuum of disjunction:

$$(A \wedge B) \vdash C \text{ iff } A \vdash (B \rightarrow C) \text{ iff } B \vdash (A \rightarrow C),$$
$$C \vdash (A \vee B) \text{ iff } (C \prec A) \vdash B \text{ iff } (C \prec B) \vdash A.$$

The strong negation $\sim$ is a *primitive* negation. Other kinds of negation connectives are *definable* in the presence of $\rightarrow$ and $\prec$. Let p be a certain propositional letter. Then we define non-falsity as follows: $\top := (p \rightarrow p)$, and non-truth in this way: $\bot := (p \prec p)$. We can then introduce two negation connectives:

$$-A := (\top \prec A) \text{ (co-negation), and}$$
$$\neg A := (A \rightarrow \bot) \text{ (intuitionistic negation).}$$

Other defined connectives of HB are equivalence, $\leftrightarrow$, and co-equivalence, $\succ\!\prec$, which are defined as follows:

$$A \equiv B := (A \rightarrow B) \wedge (B \rightarrow A); \quad A \succ\!\prec B := (A \prec B) \vee (B \prec A).$$

The connectives $\wedge, \vee, \rightarrow$, and $\prec$ are the primitive connectives of bi-intuitionistic logic BiInt, also known as Heyting-Brouwer logic HB or as subtractive logic, see [5,6,10,11,12,16,28,29,30,33,52]. Extensions of HB by strong negation $\sim$ have been introduced and investigated in [48], see also [18]. Logics with strong negation and intuitionistic implication have been introduced by David Nelson in the late 1940s and subsequently have been investigated by many researchers, see, for example, [1,8,14,15,17,19,21,22,23,25,35,37,38,39,40,42,44,46].

2 Syntax and relational semantics of HB

The propositional language $\mathcal{L}'$ of HB is defined in Backus–Naur form as follows:

$$\begin{aligned} \text{atomic formulas: } & p \in \mathit{Atom} \\ \text{formulas: } & A \in \mathit{Form(Atom)} \\ & A ::= p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (A \prec A). \end{aligned}$$

It is well-known that intuitionistic propositional logic is faithfully embeddable into the modal logic S4 (= KT4), the logic of necessity and possibility on reflexive and transitive frames. The relational frame semantics of HB is simple and reveals that HB can be faithfully embedded into temporal S4 (= $\mathsf{K}_t\mathsf{T4}$).

Definition 2.1 A frame is a pre-order $\langle I, \leq \rangle$. Intuitively, I is a non-empty set of information states, and $\leq$ is a reflexive transitive binary relation of possible expansion of states on I.

Instead of $w \leq w'$, we also write $w' \geq w$.

Definition 2.2 An HB-model is a structure $\langle I, \leq, v^+ \rangle$, where $\langle I, \leq \rangle$ is a frame and v^+ is a function that maps every $p \in \mathit{Atom}$ to a subset of I (namely the states that support the truth of p). It is assumed that v^+ satisfies the following persistence (or heredity) condition for atoms:

$$\text{if } w \leq w', \text{ then } w \in v^+(p) \text{ implies } w' \in v^+(p).$$

The relation $\mathcal{M}, w \models^+ A$ ('state w supports the truth of $\mathcal{L}'$-formula A in model $\mathcal{M}$') is inductively defined as follows:

$$\begin{aligned} \mathcal{M}, w \models^+ p \quad & \text{iff } w \in v^+(p) \\ \mathcal{M}, w \models^+ (A \wedge B) \quad & \text{iff } \mathcal{M}, w \models^+ A \text{ and } \mathcal{M}, w \models^+ B \\ \mathcal{M}, w \models^+ (A \vee B) \quad & \text{iff } \mathcal{M}, w \models^+ A \text{ or } \mathcal{M}, w \models^+ B \\ \mathcal{M}, w \models^+ (A \to B) \quad & \text{iff for every } w' \geq w : \mathcal{M}, w' \not\models^+ A \text{ or } \mathcal{M}, w' \models^+ B \\ \mathcal{M}, w \models^+ (A \prec B) \quad & \text{iff there exists } w' \leq w : \mathcal{M}, w' \models^+ A \text{ and } \mathcal{M}, w' \not\models^+ B \end{aligned}$$

where $\mathcal{M}, w \not\models^+ A$ is the classical negation of $\mathcal{M}, w \models^+ A$.

For intuitionistic negation and co-negation one obtains the following support of truth conditions:

$$\mathcal{M}, w \models^+ \neg A \text{ iff for every } w' \geq w,\ \mathcal{M}, w' \not\models^+ A;$$

$$\mathcal{M}, w \models^+ -A \text{ iff there exists } w' \leq w \text{ and } \mathcal{M}, w' \not\models^+ A.$$

Proposition 2.3 *For every $\mathcal{L}'$-formula A, HB-model $\langle I, \leq, v^+ \rangle$, and w, $w' \in I$:*

$$\textit{if } w \leq w',\ \textit{then } \mathcal{M}, w \models^+ A \textit{ implies } \mathcal{M}, w' \models^+ A.$$

Definition 2.4 HB is the set of all $\mathcal{L}'$-formulas A such that for every HB-model $\langle I, \leq, v^+ \rangle$, and $w \in I$: $\mathcal{M}, w \models^+ A$.

3 Extensions of HB by strong negation

The propositional language $\mathcal{L}$ is defined in Backus–Naur form as follows:

atomic formulas: $p \in Atom$

formulas: $A \in Form(Atom)$

$$A ::= p \mid {\sim}A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \prec A).$$

Definition 3.1 A model is a structure $\langle I, \leq, v^+, v^- \rangle$, where $\langle I, \leq \rangle$ is a frame. Moreover, v^+ and v^- are functions that map every $p \in Atom$ to a subset of I (namely the states that support the truth of p and the falsity of p, respectively. The functions v^+ and v^- satisfy the following persistence conditions for atoms: if $w \leq w'$, then $w \in v^+(p)$ implies $w' \in v^+(p)$; if $w \leq w'$, then $w \in v^-(p)$ implies $w' \in v^-(p)$. The relations $\mathcal{M}, w \models^+ A$ ('state w supports the truth of $\mathcal{L}$-formula A in model $\mathcal{M}$') and $\mathcal{M}, w \models^- A$ ('state w supports the falsity of $\mathcal{L}$-formula A in model $\mathcal{M}$') are inductively defined as follows:

$$\begin{array}{ll}
\mathcal{M}, w \models^+ p & \text{iff } w \in v^+(p) \\
\mathcal{M}, w \models^- p & \text{iff } w \in v^-(p) \\
\mathcal{M}, w \models^+ {\sim}A & \text{iff } \mathcal{M}, w \models^- A \\
\mathcal{M}, w \models^- {\sim}A & \text{iff } \mathcal{M}, w \models^+ A \\
\mathcal{M}, w \models^+ (A \wedge B) & \text{iff } \mathcal{M}, w \models^+ A \text{ and } \mathcal{M}, w \models^+ B \\
\mathcal{M}, w \models^- (A \wedge B) & \text{iff } \mathcal{M}, w \models^- A \text{ or } \mathcal{M}, w \models^- B \\
\mathcal{M}, w \models^+ (A \vee B) & \text{iff } \mathcal{M}, w \models^+ A \text{ or } \mathcal{M}, w \models^+ B \\
\mathcal{M}, w \models^- (A \vee B) & \text{iff } \mathcal{M}, w \models^- A \text{ and } \mathcal{M}, w \models^- B \\
\mathcal{M}, w \models^+ (A \rightarrow B) & \text{iff for every } w' \geq w : \mathcal{M}, w' \not\models^+ A \text{ or } \mathcal{M}, w' \models^+ B \\
\mathcal{M}, w \models^+ (A \prec B) & \text{iff there exists } w' \leq w : \mathcal{M}, w' \models^+ A \text{ and } \mathcal{M}, w' \not\models^+ B.
\end{array}$$

In Table 4, a number of natural support of falsity conditions for strongly negated implications and co-implications are listed. For each choice of pairs of conditions, support of falsity is persistent for arbitrary formulas.

cI_1	$\mathcal{M}, w \models^- (A \to B)$	iff	$\mathcal{M}, w \models^+ A$ and $\mathcal{M}, w \models^- B$
cI_2	$\mathcal{M}, w \models^- (A \to B)$	iff	for every $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^- B$
cI_3	$\mathcal{M}, w \models^- (A \to B)$	iff	there is $w' \leq w : \mathcal{M}, w' \models^+ A$ and $\mathcal{M}, w' \not\models^+ B$
cI_4	$\mathcal{M}, w \models^- (A \to B)$	iff	there is $w' \leq w : \mathcal{M}, w' \not\models^- A$ and $\mathcal{M}, w' \models^- B$
cC_1	$\mathcal{M}, w \models^- (A \prec B)$	iff	$\mathcal{M}, w \models^- A$ or $\mathcal{M}, w \models^+ B$
cC_2	$\mathcal{M}, w \models^- (A \prec B)$	iff	there is $w' \leq w : \mathcal{M}, w' \models^- A$ and $\mathcal{M}, w' \not\models^+ B$
cC_3	$\mathcal{M}, w \models^- (A \prec B)$	iff	for every $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^+ B$
cC_4	$\mathcal{M}, w \models^- (A \prec B)$	iff	for every $w' \geq w : \mathcal{M}, w' \models^- A$ or $\mathcal{M}, w' \not\models^- B$

Table 4
Support of falsity conditions for implications and co-implications

Proposition 3.2 *For every $\mathcal{L}$-formula A, model $\langle I, \leq, v^+, v^- \rangle$, and w, $w' \in I$: if $w \leq w'$ then $w \models^+ A$ implies $w' \models^+ A$; if $w \leq w'$, then $w \models^- A$ implies $w' \models^- A$.*

The different support of falsity conditions for implications and co-implications listed in Table 4 result in *sixteen* systems of constructive propositional logic with strong negation that extend HB. Valid equivalences characteristic of these logics are stated in Table 5. The logics in the language $\mathcal{L}$ that differ from each other only with respect to validating a certain pair of these equivalences (one from the I-equivalences and one from the C-equivalences) are referred to as systems (I_i, C_j), $i, j \in \{1, 2, 3, 4\}$.[2]

I_1	$\sim(A \to B) \leftrightarrow (A \wedge \sim B)$	negated implication, classical reading
I_2	$\sim(A \to B) \leftrightarrow (A \to \sim B)$	negated implication, connexive reading
I_3	$\sim(A \to B) \leftrightarrow (A \prec B)$	negated implication as co-implication
I_4	$\sim(A \to B) \leftrightarrow (\sim B \prec \sim A)$	negated implication as contraposed co-impl.
C_1	$\sim(A \prec B) \leftrightarrow (\sim A \vee B)$	negated co-implication, classical reading
C_2	$\sim(A \prec B) \leftrightarrow (\sim A \prec B)$	negated co-implication, connexive reading
C_3	$\sim(A \prec B) \leftrightarrow (A \to B)$	negated co-implication as implication
C_4	$\sim(A \prec B) \leftrightarrow (\sim B \to \sim A)$	negated co-implication as contraposed impl.

Table 5
Constructively negated implications and co-implications

[2] In the sequel I will sometimes omit the qualification $i, j \in \{1, 2, 3, 4\}$.

Definition 3.3 The logics (I_i, C_j) are defined as the triples $(\mathcal{L}, \models^+_{I_i,C_j}, \models^-_{I_i,C_j})$, where the entailment relations $\models^+_{I_i,C_j}$, $\models^-_{I_i,C_j} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ are defined as follows:
$\Delta \models^+_{I_i,C_j} \Gamma$ iff for every model $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ defined with clauses cI_i and cC_j and every $w \in I$, if $\mathcal{M}, w \models^+ A$ for every $A \in \Delta$, then $\mathcal{M}, w \models^+ B$ for some $B \in \Gamma$, and
$\Delta \models -_{I_i,C_j} \Gamma$ iff for every model $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ defined with clauses cI_i and cC_j and every $w \in I$, if $\mathcal{M}, w \models^- A$ for every $A \in \Gamma$, then $\mathcal{M}, w \models^- B$ for some $B \in \Delta$.
For singleton sets $\{A\}$ and $\{B\}$, we write $A \models^+_{I_i,C_j} B$ ($A \models^-_{I_i,C_j} B$) instead of $\{A\} \models^+_{I_i,C_j} \{B\}$ ($\{A\} \models^-_{I_i,C_j} \{B\}$).

This definition of a logic as comprising *two* entailment relations instead of just one is unusual but not at all unnatural, see, for instance, [34,49,50]. The set of all constructively false sentences is not the complement of the set of all constructively true sentences, and we can make the following observation.

Proposition 3.4 *If* $(I_i, C_j) \neq (I_4, C_4)$, *then* $\models^+_{I_i,C_j} \neq \models^-_{I_i,C_j}$.

We do not require that for atomic formulas p, $v^+(p) \cap v^-(p) = \varnothing$. Therefore, the logics under consideration are *paraconsistent*. Neither is it the case that for any formula B, $\{p, {\sim}p\} \models^+_{I_i,C_j} B$ nor is it the case that $B \models^-_{I_i,C_j} \{p, {\sim}p\}$.[3]

The next observation on negation normal forms is used in the proof of the completeness result in Section 5. A formula is in *negation normal form* (nnf) if it contains $\sim$ only in front of atoms. The following translations ρ_{I_i,C_j} send every formula A to a formula in nnf, where $p \in Atom$ and $\odot \in \{\vee, \wedge, \to, \prec\}$:

$$
\begin{aligned}
\rho_{I_i,C_j}(p) &= p \\
\rho_{I_i,C_j}({\sim}p) &= {\sim}p \\
\rho_{I_i,C_j}({\sim}{\sim} A) &= \rho_{I_i,C_j}(A) \\
\rho_{I_i,C_j}(A \odot B) &= \rho_{I_i,C_j}(A) \odot \rho_{I_i,C_j}(B) \\
\rho_{I_i,C_j}({\sim}(A \vee B)) &= \rho_{I_i,C_j}({\sim}A) \wedge \rho_{I_i,C_j}({\sim}B) \\
\rho_{I_i,C_j}({\sim}(A \wedge B)) &= \rho_{I_i,C_j}({\sim}A) \vee \rho_{I_i,C_j}({\sim}B) \\
\rho_{I_1,C_j}({\sim}(A \to B)) &= \rho_{I_1,C_j}(A) \wedge \rho_{I_1,C_j}({\sim}B) \\
\rho_{I_2,C_j}({\sim}(A \to B)) &= \rho_{I_2,C_j}(A) \to \rho_{I_2,C_j}({\sim}B) \\
\rho_{I_3,C_j}({\sim}(A \to B)) &= \rho_{I_3,C_j}(A) \prec \rho_{I_3,C_j}(B) \\
\rho_{I_4,C_j}({\sim}(A \to B)) &= \rho_{I_4,C_j}({\sim}B) \prec \rho_{I_4,C_j}({\sim}A) \\
\rho_{I_i,C_1}({\sim}(A \prec B)) &= \rho_{I_i,C_1}({\sim}A) \vee \rho_{I_i,C_1}(B) \\
\rho_{I_i,C_2}({\sim}(A \prec B)) &= \rho_{I_i,C_2}({\sim}A) \prec \rho_{I_i,C_2}(B) \\
\rho_{I_i,C_3}({\sim}(A \prec B)) &= \rho_{I_i,C_3}(A) \to \rho_{I_i,C_3}(B) \\
\rho_{I_i,C_4}({\sim}(A \prec B)) &= \rho_{I_i,C_4}({\sim}B) \to \rho_{I_i,C_4}({\sim}A)
\end{aligned}
$$

[3] Co-negation is, of course, also a paraconsistent negation, see [4,36], whereas intuitionistic negation is paracomplete, i.e., does not validate the law of excluded middle.

Lemma 3.5 *For every formula A, $\rho_{I_i,C_j}(A)$ is in negation normal form and $A \models^+_{I_i,C_j} \rho_{I_i,C_j}(A)$, $\rho_{I_i,C_j}(A) \models^+_{I_i,C_j} A$, $A \models^-_{I_i,C_j} \rho_{I_i,C_j}(A)$, $\rho_{I_i,C_j}(A) \models^-_{I_i,C_j} A$.*

4 Inferentialist (proof-theoretic) interpretation

The plan now is to interpret the connectives of $\mathcal{L}$ in the style of the Brouwer-Heyting-Kolmogorov (BHK) interpretation of the intuitionistic connectives in terms of canonical proofs, see, for example, [7, p. 154]. It is well-known that David Nelson's constructive logics with strong negation admit of a sound interpretation in terms of both proofs and disproofs, see [20,40]. We will supplement the BHK interpretation by interpretations in terms of canonical disproofs, canonical reductions to absurdity (alias non-truth), and canonical reductions to non-falsity. That is, we will define the notions of canonical proofs, disproofs, dual proofs, and dual disproofs of complex $\mathcal{L}$-formulas by simultaneous induction. To make sure that the interpretation is correct for the logics (I_i, C_j), we will make the following assumptions:

(i) for no $\mathcal{L}$-formula A there exists both a proof and a dual proof of A;

(ii) for no $\mathcal{L}$-formula A there exists both a disproof and a dual disproof of A;

(iii) every $\mathcal{L}$-formula A either has a proof or dual proof;

(iv) every $\mathcal{L}$-formula A either has a disproof or dual disproof.

Note that we do not need clauses for the constants $\bot$ and $\top$ and the negation operations $\neg$ and $-$, because in $\mathcal{L}$ these connectives are definable. We also assume that the conditions under which an entity is a canonical proof, disproof, dual proof, or dual disproof of an atomic sentence depend on the appropriate and relevant social practice and are not a matter of logic.

4.1 Canonical proofs

We first consider the inductive definition of the notion of a canonical proof of a compound $\mathcal{L}$-formula.

- A canonical proof of a strongly negated formula $\sim A$ is a canonical disproof of A.
- A canonical proof of a conjunction $(A \wedge B)$ is a pair (π_1, π_2) consisting of a canonical proof π_1 of A and a canonical proof π_2 of B.
- A canonical proof of a disjunction $(A \vee B)$ is a pair (i, π) such that $i = 0$ and π is a canonical proof of A or $i = 1$ and π is a canonical proof of B.
- A canonical proof of an implication $(A \to B)$ is a construction that transforms any canonical proof of A into a canonical proof of B.
- A canonical proof of a co-implication $(A \prec B)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (This pair is a canonical dual proof of $(A \to B)$).

4.2 Canonical disproofs

- A canonical disproof of a strongly negated formula $\sim A$ is a canonical proof of A.
- A canonical disproof of a conjunction $(A \wedge B)$ is a pair (i, π) such that $i = 0$ and π is a canonical disproof of A or $i = 1$ and π is a canonical disproof of B.
- A canonical disproof of a disjunction $(A \vee B)$ is a pair (π_1, π_2) consisting of a canonical disproof π_1 of A and a canonical disproof π_2 of B.
- A canonical disproof of an implication $(A \to B)$ in
  (I_1C_j) is a pair (π_1, π_2) consisting of a canonical proof π_1 of A and a canonical disproof π_2 of B.
  (I_2C_j) is a construction that transforms any canonical proof of A into a canonical disproof of B.
  (I_3C_j) is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (This pair is a canonical dual proof of $(A \to B)$).
  (I_4C_j) is a pair (π_1, π_2), where π_1 is a canonical disproof of B and π_2 is a canonical dual disproof of A.
- A canonical disproof of a co-implication $(A \prec B)$ in
  (I_iC_1) is a pair (i, π) such that $i = 0$ and π is a canonical disproof of A or $i = 1$ and π is a canonical proof of B.
  (I_iC_2) is a pair (π_1, π_2), where π_1 is a canonical disproof of A and π_2 is a canonical dual proof of B. (This pair is a canonical dual proof of $(\sim A \to B)$).
  (I_iC_3) is a construction that transforms any canonical proof of A into a canonical proof of B.
  (I_iC_4) is a construction that transforms any canonical disproof of B into a canonical disproof of A.

4.3 Canonical reductions to non-truth (canonical dual proofs)

- A canonical reduction to non-truth of a strongly negated formula $\sim A$ is canonical dual disproof of A.
- A canonical reduction to non-truth of a conjunction $(A \wedge B)$ is a pair (i, π) such that $i = 0$ and π is a canonical dual proof of A or $i = 1$ and π is a canonical dual proof of B.
- A canonical reduction to non-truth of a disjunction $(A \vee B)$ is a pair (π_1, π_2) consisting of a dual proof π_1 of A and a dual proof π_2 of B.
- A canonical reduction to non-truth of an implication $(A \to B)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (This pair is a canonical proof of $(A \prec B)$).
- A canonical reduction to non-truth of a co-implication $(A \prec B)$ is a construction that transforms any dual proof of B into a dual proof of A.

4.4 Canonical reductions to non-falsity (canonical dual disproofs)

- A canonical reduction to non-falsity of a strongly negated formula $\sim A$ is a canonical dual proof of A.
- A canonical reduction to non-falsity of a conjunction $(A \wedge B)$ is a pair (π_1, π_2) consisting of a dual disproof π_1 of A and a dual disproof π_2 of B.
- A canonical reduction to non-falsity of a disjunction $(A \vee B)$ is a pair (i, π) such that $i = 0$ and π is a canonical dual disproof of A or $i = 1$ and π is a canonical dual disproof of B.
- A canonical reduction to non-falsity of an implication $(A \rightarrow B)$ in
  $(I_1 C_j)$ is a pair (i, π) such that $i = 0$ and π is a canonical dual proof of A or $i = 1$ and π is a canonical dual disproof of B.
  $(I_2 C_j)$ is a pair (π_1, π_s), where pi_1 is a canonical proof of A and π_2 is a canonical dual disproof of B.
  $(I_3 C_j)$ is a construction that transforms any canonical dual proof of B into a canonical dual proof of A. (This pair is a canonical dual proof of $(A \prec B)$).
  $(I_4 C_j)$ is a construction that transforms any canonical dual disproof of A into a canonical dual disproof of B.
- A canonical reduction to non-falsity of a co-implication $(A \prec B)$ in
  $(I_i C_1)$ is a pair (π_1, π_2), where π_1 is a canonical dual disproof of A and π_2 is a canonical dual proof of B.
  $(I_i C_2)$ is a construction that transforms any canonical dual proof of B into a canonical dual disproof of A. (This construction is a canonical dual proof of $(\sim A \prec B)$).
  $(I_i C_3)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (This pair is a canonical dual proof of $(A \rightarrow B)$).
  $(I_i C_4)$ is a pair (π_1, π_2), where π_1 is a canonical disproof of B and π_2 is a canonical dual disproof of A.

In order to show by induction on the construction of inferences that the logics (I_i, C_j) are sound with respect to the above BHK-style interpretation in terms of proofs, disproofs, and their duals, we need proof systems for the semantically defined logics (I_i, C_j). We consider the display calculi defined in [48].

5 Display calculi

The structural proof theory of bi-intuitionistic logic is confronted with a number of problems, which are described, for example, in [5,10,48]. The sequent calculus for Heyting-Brouwer logic in [6] uses single-conclusion sequents but imposes a 'singleton on the left' constraint on the left introduction rule for co-implication (and a 'singleton on the right' constraint on the right introduction rule for implication). This asymmetric sequent calculus does not enjoy cut-elimination. Also the sequent calculus for HB in [28] does not allow cut-elimination. These problems can be overcome in display logic and in other types of sequent calculi that differ from ordinary Gentzen systems, see [5,10,11,12,48]. We will use the display sequent calculi $\delta(I_i, C_j)$ for the logics (I_i, C_j)

developed [48] and therefore briefly rehearse the presentation of $\delta(I_i, C_j)$.

The set of structures (or Gentzen terms) is defined as follows:

$$\text{formulas: } A \in \mathit{Form(Atom)}$$

$$\text{structures } X \in \mathit{Struc(Form)}$$

$$X ::= A \mid \mathbf{I} \mid (X \circ X) \mid (X \bullet X).$$

The intended interpretation of the connective $\circ$ as conjunction in antecedent position and as implication in succedent position and of $\bullet$ as co-implication in antecedent position and as disjunction in succedent position justifies certain 'display postulates' (dp) (we omit outer brackets, each column states two structural inference rules):

$$\frac{\dfrac{Y \vdash X \circ Z}{X \circ Y \vdash Z}}{X \vdash Y \circ Z} \qquad \frac{\dfrac{X \vdash Y \circ Z}{X \circ Y \vdash Z}}{Y \vdash X \circ Z} \qquad \frac{\dfrac{X \bullet Z \vdash Y}{X \vdash Y \bullet Z}}{X \bullet Y \vdash Z} \qquad \frac{\dfrac{X \bullet Y \vdash Z}{X \vdash Y \bullet Z}}{X \bullet Z \vdash Y}$$

Moreover, the interpretation of $\mathbf{I}$ as the empty structure suggests the following structural inference rules:

$$\frac{\dfrac{X \circ \mathbf{I} \vdash Y}{X \vdash Y}}{\mathbf{I} \circ X \vdash Y} \qquad \frac{\dfrac{\mathbf{I} \circ X \vdash Y}{X \vdash Y}}{X \circ \mathbf{I} \vdash Y} \qquad \frac{\dfrac{X \vdash Y \bullet \mathbf{I}}{X \vdash Y}}{X \vdash \mathbf{I} \bullet Y} \qquad \frac{\dfrac{X \vdash \mathbf{I} \bullet Y}{X \vdash Y}}{X \vdash Y \bullet \mathbf{I}}$$

In addition there are various 'logical' structural rules:

$$\frac{}{p \vdash p}\ (id) \qquad \frac{}{\sim p \vdash \sim p}\ (id\sim) \qquad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y}\ (cut)$$

and versions of the familiar structural rules from standard Gentzen systems for classical logic, monotonicity (weakening), exchange (permutation), and contraction, together with associativity, see Table 6.

$$\frac{X \vdash Y}{X \vdash Y \bullet Z}\ (rm) \qquad \frac{X \vdash Y}{X \circ Z \vdash Y}\ (lm)$$

$$\frac{X \vdash Y \bullet Z}{X \vdash Z \bullet Y}\ (re) \qquad \frac{X \circ Z \vdash Y}{Z \circ X \vdash Y}\ (le)$$

$$\frac{X \vdash Y \bullet Y}{X \vdash Y}\ (rc) \qquad \frac{X \circ X \vdash Y}{X \vdash Y}\ (lc)$$

$$\frac{X \vdash (Y \bullet Z) \bullet X'}{X \vdash Y \bullet (Z \bullet X')}\ (ra) \qquad \frac{(X \circ Y) \circ Z \vdash X'}{X \circ (Y \circ Z) \vdash X'}\ (la)$$

Table 6
Structural sequent rules

$$\frac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash (A \wedge B)}\ (\vdash \wedge) \qquad \frac{A \circ B \vdash X}{(A \wedge B) \vdash X}\ (\wedge \vdash)$$

$$\frac{X \vdash A \bullet B}{X \vdash (A \vee B)}\ (\vdash \vee) \qquad \frac{A \vdash X \quad B \vdash Y}{(A \vee B) \vdash X \bullet Y}\ (\vee \vdash)$$

$$\frac{X \vdash A \circ B}{X \vdash (A \to B)}\ (\vdash \to) \qquad \frac{X \vdash A \quad B \vdash Y}{(A \to B) \vdash X \circ Y}\ (\to \vdash)$$

$$\frac{X \vdash B \quad A \vdash Y}{X \bullet Y \vdash B \prec A}\ (\vdash \prec) \qquad \frac{B \bullet A \vdash X}{B \prec A \vdash X}\ (\prec \vdash)$$

$$\frac{X \vdash {\sim}A \bullet {\sim}B}{X \vdash {\sim}(A \wedge B)}\ (\vdash {\sim}\wedge) \qquad \frac{{\sim}A \vdash X \quad {\sim}B \vdash Y}{{\sim}(A \wedge B) \vdash X \bullet Y}\ ({\sim}\wedge \vdash)$$

$$\frac{X \vdash {\sim}A \quad Y \vdash {\sim}B}{X \circ Y \vdash {\sim}(A \vee B)}\ (\vdash {\sim}\vee) \qquad \frac{{\sim}A \circ {\sim}B \vdash X}{{\sim}(A \vee B) \vdash X}\ ({\sim}\vee \vdash)$$

$$\frac{X \vdash A}{X \vdash {\sim}{\sim}A}\ (\vdash {\sim}{\sim}) \qquad \frac{A \vdash X}{{\sim}{\sim}A \vdash X}\ ({\sim}{\sim} \vdash)$$

Table 7
Introduction rules shared by all logics (I_i, C_j)

The display sequent calculi $\delta(I_i, C_j)$, $i, j \in \{1, 2, 3, 4\}$, for the constructive logics (I_i, C_j) share these rules and the introduction rules stated in Table 7. The particular display calculus $\delta(I_i, C_j)$ then is the proof system obtained by adding the rules rI_i and rC_j from Table 8.

A derivation of a sequent s from a set of sequents $\{\mathsf{s}_1, \ldots, \mathsf{s}_n\}$ in $\delta(I_i, C_j)$ is defined as a tree with root s such that every leaf is an instantiation of (id), $(id{\sim})$, or a sequent from $\{\mathsf{s}_1, \ldots, \mathsf{s}_n\}$, and every other node is obtained by an application of one of the remaining rules. A proof of a sequent s in $\delta(I_i, C_j)$ is a derivation of s from $\varnothing$. Sequents s and s' are said to be interderivable iff s is derivable from $\{\mathsf{s}'\}$ and s' is derivable from $\{\mathsf{s}\}$.

Two sequents s and s' are said to be structurally equivalent if they are interderivable by means of display postulates only. It is characteristic for display calculi that any substructure of a given sequent s may be displayed as the entire antecedent or succedent of a structurally equivalent sequent s'.

If $\mathsf{s} = X \vdash Y$ is a sequent, then the displayed occurrence of X (Y) is an antecedent (succedent) part of s. If an occurrence of $(Z \circ W)$ is an antecedent part of s, then the displayed occurrences of Z and W are antecedent parts of s. If an occurrence of $(Z \bullet W)$ is an antecedent part of s, then the displayed occurrence of Z (W) is an antecedent (succedent) part of s. If an occurrence of $(Z \circ W)$ is a succedent part of s, then the displayed occurrence of Z (W) is an antecedent (succedent) part of s. If an occurrence

rI_1	$\dfrac{X \vdash A \quad Y \vdash \sim B}{X \circ Y \vdash \sim(A \to B)}$	$\dfrac{A \circ \sim B \vdash X}{\sim(A \to B) \vdash X}$
rI_2	$\dfrac{X \vdash A \circ \sim B}{X \vdash \sim(A \to B)}$	$\dfrac{X \vdash A \quad \sim B \vdash Y}{\sim(A \to B) \vdash X \circ Y}$
rI_3	$\dfrac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash \sim(A \to B)}$	$\dfrac{A \bullet B \vdash X}{\sim(A \to B) \vdash X}$
rI_4	$\dfrac{X \vdash \sim B \quad \sim A \vdash Y}{X \bullet Y \vdash \sim(A \to B)}$	$\dfrac{\sim B \bullet \sim A \vdash X}{\sim(A \to B) \vdash X}$
rC_1	$\dfrac{X \vdash \sim A \bullet B}{X \vdash \sim(A \mathrel{-\!\!\!<} B)}$	$\dfrac{\sim A \vdash X \quad B \vdash Y}{\sim(A \mathrel{-\!\!\!<} B) \vdash X \bullet Y}$
rC_2	$\dfrac{X \vdash \sim A \quad B \vdash Y}{X \bullet Y \vdash \sim(A \mathrel{-\!\!\!<} B)}$	$\dfrac{\sim A \bullet B \vdash X}{\sim(A \mathrel{-\!\!\!<} B) \vdash X}$
rC_3	$\dfrac{X \vdash A \circ B}{X \vdash \sim(A \mathrel{-\!\!\!<} B)}$	$\dfrac{Y \vdash A \quad B \vdash X}{\sim(A \mathrel{-\!\!\!<} B) \vdash Y \circ X}$
rC_4	$\dfrac{X \vdash \sim B \circ \sim A}{X \vdash \sim(A \mathrel{-\!\!\!<} B)}$	$\dfrac{Y \vdash \sim B \quad \sim A \vdash X}{\sim(A \mathrel{-\!\!\!<} B) \vdash Y \circ X}$

Table 8
Sequent rules for negated implications and co-implications

of $(Z \bullet W)$ is a succedent part of s, then the displayed occurrences of Z and W are succedent parts of s.

Theorem 5.1 (cf. (Belnap 1982)) *For every sequent* s *and every antecedent (succedent) part* X *of* s*, there exists a sequent* s' *structurally equivalent to* s *such that* X *is the entire antecedent (succedent) of* s'*.*

Proposition 5.2 *For every* $\mathcal{L}$*-formula* A *and every calculus* $\delta(I_i, C_j)$*,* $A \vdash A$ *is provable.*

One can define translations τ_1 and τ_2 from structures into formulas such that these translations make explicit the intuitive, context-sensitive interpretation of the structural connectives: translation τ_1 translates structures which are antecedent parts of a sequent, whereas τ_2 translates structures which are succedent parts of a sequent.

Definition 5.3 The translations τ_1 and τ_2 from structures into formulas are inductively defined as follows, where A is a formula and p is a certain atom:

$$\begin{array}{rclrcl} \tau_1(A) &=& A & \tau_2(A) &=& A \\ \tau_1(\mathbf{I}) &=& p \rightarrow p & \tau_2(\mathbf{I}) &=& p \prec p \\ \tau_1(X \circ Y) &=& \tau_1(X) \wedge \tau_1(Y) & \tau_2(X \circ Y) &=& \tau_1(X) \rightarrow \tau_2(Y) \\ \tau_1(X \bullet Y) &=& \tau_1(X) \prec \tau_2(Y) & \tau_2(X \bullet Y) &=& \tau_2(X) \vee \tau_2(Y) \end{array}$$

Theorem 5.4 (Soundness) *(1) If the $X \vdash Y$ is provable in $\delta(I_i, C_j)$, then $\tau_1(X) \models^+_{I_i,C_j} \tau_2(Y)$. (2) If $X \vdash Y$ is provable in $\delta(I_i, C_j)$, then $\sim\tau_2(Y) \models^-_{I_i,C_j} \sim\tau_1(X)$.*

The language $\mathcal{L}^*$ results from $\mathcal{L}$ by adding for every atomic formula p a new atom p^*. If A is an $\mathcal{L}$-formula, $(A)^*$ is the result of replacing every strongly negated atom $\sim p$ in A by p^*.

Lemma 5.5 *For every $\mathcal{L}$-formula A, if $\varnothing \models^+_{I_i,C_j} A$, then $(\rho_{I_i,C_j}(A))^*$ is valid in* HB.

Lemma 5.6 *For every $\sim$-free $\mathcal{L}$-formula A, if A is provable in* HB*, then $\mathbf{I} \vdash A$ is provable in $\delta(I_i, C_j)$ without using any sequent rules for strongly negated formulas.*

Lemma 5.7 *For every $\mathcal{L}$-formula A, $A \vdash \rho_{I_i,C_j}(A)$ and $\rho_{I_i,C_j}(A) \vdash A$ are provable in $\delta(I_i, C_j)$.*

Lemma 5.8 *Every sequent $X \vdash \tau_1(X)$ and $\tau_2(X) \vdash X$ is provable in $\delta(I_i, C_j)$, for all $i, j \in \{1, 2, 3, 4\}$.*

Theorem 5.9 (Completeness) *(1) If $\rho_{I_i,C_j}(\tau_1(X)) \models^+_{I_i,C_j} \rho_{I_i,C_j}(\tau_2(Y))$, then $X \vdash Y$ is provable in $\delta(I_i, C_j)$. (2) If $\rho_{I_i,C_j}(\sim\tau_2(Y)) \models^-_{I_i,C_j} \rho_{I_i,C_j}(\sim\tau_1(X))$, then $X \vdash Y$ is provable in $\delta(I_i, C_j)$.*

Let $\delta(I_i, C_j)^+$ denote the result of dropping all sequent rules exhibiting $\sim$ from $\delta(I_i, C_j)$.

Theorem 5.10 *If $X \vdash Y$ is provable in system $\delta(I_i, C_j)$, then $(\rho_{Ii,C_j}(\tau_1(X)))^* \vdash (\rho_{Ii,C_j}(\tau_2(Y)))^*$ is provable in $\delta(I_i, C_j)^+$ without any applications of (cut).*

6 Correctness of the the logics (I_i, C_j) with respect to the inferentialist semantics

We show that if a sequent is provable in $\delta(I_i, C_j)$, then there exists a certain construction made up from proofs, disproofs, and their duals that transforms any proof of the antecedent of the sequent into a proof of its succedent. [4]

Theorem 6.1 *Let $i, j \in \{1, 2, 3, 4\}$. If $X \vdash Y$ is provable in $\delta(I_i, C_j)$, then*

(i) *there exists a construction π such that $\pi(\pi')$ is a canonical proof of $\tau_2(Y)$ whenever π' is a canonical proof of $\tau_1(X)$.*

[4] This result may give rise to a four-sorted typed λ-calculus.

(ii) *there exists a construction π such that $\pi(\pi')$ is a canonical dual proof of $\tau_1(X)$ whenever π' is a canonical dual proof of $\tau_2(Y)$.*

Proof. By simultaneous induction on derivations in $\delta(I_i, C_j)$.
(i): We first consider the display postulates. The first display postulates for $\circ$ are:

$$\frac{\dfrac{Y \vdash X \circ Z}{X \circ Y \vdash Z}}{X \vdash Y \circ Z}$$

Suppose, by the induction hypothesis for (i), that there exists a construction π that transforms any canonical proof of $\tau_1(Y)$ into a canonical proof of $\tau_2(X \circ Z)$ $(= \tau_1(X) \rightarrow \tau_2(Z))$, i.e., into a construction that transforms any canonical proof of $\tau_1(X)$ into a canonical proof of $\tau_2(Z)$. Let π' be any canonical proof of $\tau_1(X \circ Y)$ $(= \tau_1(X) \wedge \tau_1(Y))$. The proof π' is a pair (π'_1, π'_2), where π'_1 is a canonical proof of $\tau_1(X)$ and π'_2 is canonical proof of $\tau_1(Y)$. Then $(\pi(\pi'_2))(\pi'_1)$ is a proof[5] of $\tau_2(Z)$.
Suppose next, by the induction hypothesis for (i), that there is a construction π that transforms any proof of $\tau_1(X \circ Y)$ $(= \tau_1(X) \wedge \tau_1(Y))$ into a proof of $\tau_2(Z)$. Let π' be any proof of $\tau_1(X)$. Then $\pi^*(\pi') = \pi((\pi', \ ))$ is a construction that transforms any proof of $\tau_1(Y)$ into a proof of $\tau_2(Z)$.
The second pair of display postulates for $\circ$ is dealt with similarly.
The first pair of display postulates for $\bullet$ is:

$$\frac{\dfrac{X \bullet Z \vdash Y}{X \vdash Y \bullet Z}}{X \bullet Y \vdash Z}$$

Suppose, by the induction hypothesis for (i), that there exists a construction π that transforms any proof of $\tau_1(X \bullet Z)$, $(= \tau_1(X) \prec \tau_2(Z))$, i.e., any pair (π_1, π_2), where π_1 is a proof of $\tau_1(X)$ and π_2 is a dual proof of $\tau_2(Z)$, into a proof of $\tau_2(Y)$. Let π' be any proof of $\tau_1(X)$. There either is a dual proof of $\tau_2(Z)$ or there is not. If there is such a dual proof, let π'' be a fixed dual proof of $\tau_2(Z)$. Then $(0, \pi((\pi', \pi'')))$ is a proof of $(\tau_2(Y) \vee \tau_2(Z))$. If there does not exist any dual proof of $\tau_2(Z)$, then there exists a proof of $\tau_2(Z)$. Let π''' be such a proof. Then $(1, \pi''')$ is a proof of $(\tau_2(Y) \vee \tau_2(Z))$.
Suppose now, by the induction hypothesis for (i), that there is a construction π that transforms any proof of $\tau_1(X)$ into a proof of $\tau_2(Y \bullet Z)$ $(= \tau_2(Y) \vee \tau_2(Z))$. Let π' be any proof of $\tau_1(X \bullet Y)$ $(= \tau_1(X) \prec \tau_2(Y))$, i.e., any pair (π_1, π_2), where π_1 is a proof of $\tau_1(X)$ and π_2 is a dual proof of $\tau_2(Y)$. Since $\tau_2(Y)$ has no proof, $\pi(\pi_1)$ is a proof of $\tau_2(Z)$.
The second pair of display postulates for $\bullet$ is dealt with similarly.
The case of the logical structural rules is simple; the axiomatic sequents are dealt with by the identity function and (*cut*) by functional application. The case of the other structural sequent rules from Table 6 is quite obvious.
We present here, by way of example, just the cases of three introduction rules.

[5] In the sequel I will often omit the expression 'canonical'.

($\vdash$ —<):

$$\frac{X \vdash B \quad A \vdash Y}{X \bullet Y \vdash B \prec A}$$

Suppose, by the induction hypothesis for (i), that there is a construction π that transforms any proof of $\tau_1(X)$ into a proof of $\tau_2(B)$ and, by the induction hypothesis for (ii), that there is a construction π' that transforms any dual proof of $\tau_2(Y)$ into a dual proof of $\tau_1(A)$. Let π^* be a proof of $\tau_1(X \bullet Y)$ $(= \tau_1(X) \prec \tau_2(Y))$. Then π^* is a pair (π_1, π_2), where π_1 is a proof of $\tau_1(X)$ and π_2 is a dual proof of $\tau_2(Y)$. Therefore, the pair $(\pi(\pi_1), \pi'(\pi_2))$ is proof of $(B \prec A)$.
rI_4, first rule:

$$\frac{X \vdash \sim B \qquad \sim A \vdash Y}{X \bullet Y \vdash \sim (A \rightarrow B)}$$

Suppose, by the induction hypothesis for (i), that π is a construction that transforms any proof $\tau_1(X)$ into a proof of $\sim B$, and, by the induction hypothesis for (ii), that π' is a construction that transforms any dual proof of $\tau_2(Y)$ into a dual proof of $\sim A$, i.e., into a dual disproof of A. Let π^* be a proof of $\tau_1(X \bullet Y)$ $(= \tau_1(x) \prec \tau_2(Y))$. Then π^* is a pair (π_1, π_2), where π_1 is a proof of $\tau_1(X)$ and π_2 is a dual proof of $\tau_2(Y)$. A proof of $\sim (A \rightarrow B)$ in (I_4, C_j) is a disproof of $(A \rightarrow B)$ in (I_4, C_j), which is a pair (π_1', π_2'), where π_1' is a proof of $\sim B$, and π_2' is a dual proof of $\sim A$. Note that $(\pi(\pi_1), \pi'(\pi_2))$ is such a pair.
rC_1, second rule:

$$\frac{\sim A \vdash X \quad B \vdash Y}{\sim (A \prec B) \vdash X \bullet Y}$$

Suppose, by the induction hypothesis for (i), that π' is a construction that transforms any proof of B into a proof of $\tau_2(Y)$ and that π'' is a construction that transforms any proof $\sim A$ into a proof of $\tau_2(X)$. Let π^* be a proof of $\sim (A \prec B)$ in (I_i, C_1), i.e., a disproof of $(A \prec B)$ in (I_i, C_1). Then π^* is a pair (i, π) such that $i = 0$ and π is a disproof of A or $i = 1$ and π is a proof of B. But then either $(0, \pi''(\pi))$ or $(1, \pi'(\pi))$ is a proof of $\tau_2(X \bullet Y)$.
(ii): We present here just the case of the second pair of display postulates for $\circ$:

$$\frac{\dfrac{X \vdash Y \circ Z}{X \circ Y \vdash Z}}{Y \vdash X \circ Z}$$

Suppose, by the induction hypothesis for (ii), that π is a construction that transforms any dual proof of $\tau_1(Y) \rightarrow \tau_2(Z)$ (i.e., any pair (π_1, π_2), where π_1 is a proof of $\tau_1(Y)$ and π_2 is dual proof of $\tau_2(Z)$) into a dual proof of $\tau_1(X)$. Either there is a proof of $\tau_1(Y)$ or not. If there is a proof of $\tau_1(Y)$, let π'' be such a proof. Then $(0, \pi(\pi'', \pi'))$ is a dual proof of $\tau_1(X \circ Y)$. If there is no proof of $\tau_1(Y)$, then there is a dual proof of $\tau_1(Y)$. Let π''' be such a dual proof. Then $(1, \pi''')$ is a dual proof of $\tau_1(X \circ Y)$.
Now suppose that, by the induction hypothesis for (ii), there exists a construction π that transforms any dual proof of $\tau_2(Z)$ into a dual proof of $\tau_1(x) \wedge \tau_1(Y)$. Let π' be any dual proof of $\tau_1(X) \rightarrow \tau_2(Z)$, i.e., a pair (π_1, π_2), where π_1 is a proof of $\tau_1(X)$ and

π_2 is a dual proof of $\tau_2(Z)$. Then there is no dual proof of $\tau_1(X)$ and $(0, \pi(\pi_2))$ is a dual proof of $\tau_1(Y)$. □

Corollary 6.2 *Let $i, j \in \{1, 2, 3, 4\}$. If $X \vdash Y$ is provable in $\delta(I_i, C_j)$, then*

(i) *there exists a construction π such that $\pi(\pi')$ is a canonical disproof of $\tau_2(Y)$ whenever π' is a canonical disproof of $\tau_1(X)$.*

(ii) *there exists a construction π such that $\pi(\pi')$ is a canonical dual disproof of $\tau_1(X)$ whenever π' is a canonical dual disproof of $\tau_2(Y)$.*

Proof. Every canonical disproof of A is a canonical proof of $\sim A$ and every canonical dual disproof of A is a canonical dual proof of $\sim A$. □

The following claims follow from Theorem 6.1 and Corollary 6.2.

Theorem 6.3 *Let $i, j \in \{1, 2, 3, 4\}$.*

(i) *If $\mathbf{I} \vdash A$ is provable in $\delta(I_i, C_j)$, then there exists a construction π which is a proof of A.*

(ii) *If $A \vdash \mathbf{I}$ is provable in $\delta(I_i, C_j)$, then there exists a construction π which is a dual proof of A.*

(iii) *If $\mathbf{I} \vdash \sim A$ is provable in $\delta(I_i, C_j)$, then there exists a construction π which is a disproof of A.*

(iv) *If $\sim A \vdash \mathbf{I}$ is provable in $\delta(I_i, C_j)$, then there exists a construction π which is a dual disproof of A.*

Proof. Note that any canonical proof of $\tau_1(\mathbf{I}) = (p \to p)$ and any canonical dual proof of $\tau_2(\mathbf{I}) = (p \prec p)$ is the identity function. □

Example 6.4 The sequent $\mathbf{I} \vdash q \vee -q$ is provable in the logics (I_i, C_j), and it can easily be seen that there exists a construction that is a (canonical) proof of $q \vee ((p \to p) \prec q)$. A proof of $q \vee ((p \to p) \prec q)$ is a pair (i, π), where $i = 0$ and π is a proof of q, or $i = 1$ and π is a proof of $((p \to p) \prec q)$. Now, π is a proof of $((p \to p) \prec q)$ iff π is a pair (π_1, π_2), where π_1 is a proof of $(p \to p)$ and π_2 is a dual proof of q. Since the identity function is a proof of $(p \to p)$ and since every $\mathcal{L}$-formula either has a proof or a dual proof, there exists a proof of $q \vee ((p \to p) \prec q)$.

Example 6.5 There exists a construction that is a proof of $\sim (p \to q) \to (p \prec q)$ in the logics (I_3, C_j). A proof of $\sim (p \to q) \to (p \prec q)$ in (I_3, C_j) is a construction that transforms any proof of $\sim (p \to q)$ into a proof of $(p \prec q)$. A proof of $\sim (p \to q)$ in (I_3, C_j) is a disproof of $(p \to q)$, which is a pair (π_1, π_2), where π_1 is a proof of p and π_2 is a dual proof of q. But this pair is a proof of $(p \prec q)$ in (I_3, C_j), so that the identity function is a proof of $\sim (p \to q) \to (p \prec q)$ in (I_3, C_j).

7 Summary

We have considered sixteen extensions of propositional Brouwer-Heyting logic by strong negation. Each of these logics (I_i, C_j) $(i, j \in \{1, 2, 3, 4\})$ turned out to be correct with respect to an extended BHK-style inferentialist interpretation. The interpretation makes use of four primitive notions, namely the notions of proof, disproof, dual proof, and dual disproof. This correctness result supports the view that the logics (I_i, C_j) are indeed constructive propositional logics.[6] The findings of this paper can be summarized as in a Table 9.[7]

(propositional) logic	*soundness with respect to an interpretation*
intuitionistic logic	in terms of proofs
Nelson's logics	in terms of proofs and disproofs
dual intuitionistic logic	in terms of dual proofs
bi-intuitionistic logic	in terms of proofs and dual proofs
bi-intuitionistic logic extended by strong negation	in terms of proof, disproofs, and their duals

Table 9
Summary

References

[1] A. Almukdad and D. Nelson, Constructible Falsity and Inexact Predicates, *Journal of Symbolic Logic*, 49 (1984), 231–233.

[2] N.D. Belnap, Display Logic, *Journal of Philosophical Logic* 11 (1982), 375–417. Reprinted with small modifications as §62 of A.R. Anderson, N.D. Belnap, and J.M. Dunn, *Entailment: the logic of relevance and necessity.* Vol. 2, Princeton University Press, Princeton, 1992.

[3] B. Brown, On Paraconsistency, in. L. Goble (ed.), *A Companion to Philosophical Logic*, Blackwell Publishers, Oxford, 2002, 628–650.

[4] A. Brunner and W. Carnielli, Anti-intuitionism and Paraconsistency, *Journal of Applied Logic* 3 (2005), 161-184.

[5] L. Buisman and R. Goré, A Cut-Free Sequent Calculus for Bi-intuitionistic Logic, in: N. Olivetti (ed.), *TABLEAUX 2007*, Springer Lecture Notes in AI 4548, Springer-Verlag, Berlin, 2007, 90–106.

[6] T. Crolard, Subtractive Logic, *Theoretical Computer Science* 254 (2001), 151–185.

[7] D. van Dalen, *Logic and Structure*, Springer-Verlag, Berlin, 2004.

[8] J.M. Dunn, Partiality and its Dual, *Studia Logica* 66 (2000), 5–40.

[9] D. Gabbay, What is Negation in a System, in: F. Drake and J. Truss (eds.), *Logic Colloquium '86*, Elsevier, Amsterdam, 95–112.

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[7] In intuitionistic logic, $\bot$ is primitive and has no proof; in dual intuitionistic logic, $\top$ is primitive and has no dual proof.

[10] R. Goré, Dual Intuitionistic Logic Revisited, in: Roy Dyckhoff (ed.), *TABLEAUX 2000*, Springer Lecture Notes in AI 1847, Springer-Verlag, Berlin, 2000, 252–267.

[11] R. Goré and L. Postniece, Combining Derivations and Refutations for Cut-free Completeness in Bi-intuitionistic Logic, *Journal of Logic and Computation* 20 (2010), 233–260.

[12] R. Goré, L. Postniece, and A. Tiu, Cut-elimination and Proof-search for Bi-intuitionistic Logic using Nested Sequents, in: C. Areces and R. Goldblatt (eds.), *Advances in Modal Logic. Vol. 7*, College Publications, London, 2008, 43–66.

[13] A. Grzegorczyk. A Philosophically Plausible Formal Interpretation of Intuitionistic Logic, *Indagationes Mathematicae* 26 (1964), 596-601.

[14] Y. Gurevich, Intuitionistic Logic with Strong Negation, *Studia Logica* 36 (1977), 49–59.

[15] N. Kamide, A Canonical Model Construction for Substructural Logics with Strong Negation, *Reports on Mathematical Logic* 36 (2002), 95–116.

[16] N. Kamide, A Note on Dual-intuitionistic Logic, *Mathematical Logic Quarterly* 49 (2003), 519–524.

[17] N. Kamide, Phase Semantics and Petri Net Interpretation for Resource-Sensitive Strong Negation, *Journal of Logic, Language and Information* 15 (2006), 371–401.

[18] N. Kamide and H. Wansing, Symmetric and Dual Paraconsistent Logics, 2008, to appear in: *Logic and Logical Philosophy.*

[19] M. Kracht, On Extensions of Intermediate Logics by Strong Negation, *Journal of Philosophical Logic* 27 (1998), 49–73.

[20] E.G.K. López-Escobar, Refutability and Elementary Number Theory, *Indigationes Mathematicae* 34 (1972), 362–374.

[21] D. Nelson, Constructible Falsity, *Journal of Symbolic Logic* 14 (1949), 16–26.

[22] S. Odintsov, Algebraic Semantics for Paraconsistent Nelson's Logic, *Journal of Logic and Computation* 13 (2003), 453–468.

[23] S. Odintsov, *Constructive Negations and Paraconsistency*, Springer-Verlag, Dordrecht, 2008.

[24] S. Odintsov and H. Wansing, Inconsistency-tolerant Description Logic. Motivation and Basic Systems, in: V. Hendricks and J. Malinowski (eds.), *Trends in Logic. 50 Years of Studia Logica*, Kluwer Academic Publishers, Dordrecht, 2003, 301–335.

[25] S. Odintsov and H. Wansing, Constructive Predicate Logic and Constructive Modal Logic. Formal Duality versus Semantical Duality, in: V. Hendricks et al. (eds.), *First-Order Logic Revisited*, Logos Verlag, Berlin, 2004, 269–286.

[26] S. Odintsov and H. Wansing, Inconsistency-tolerant Description Logic. Part II: Tableau Algorithms, *Journal of Applied Logic* 6 (2008), 343–360.

[27] G. Priest, *Doubt Truth to Be a Liar*, Oxford University Press, Oxford, 2006.

[28] C. Rauszer, A Formalization of the Propositional Calculus of H-B Logic, *Studia Logica* 33 (1974), 23–34.

[29] C. Rauszer, Applications of Kripke Models to Heyting-Brouwer Logic, *Studia Logica* 36 (1977), 61–72.

[30] C. Rauszer, An Algebraic and Kripke-style Approach to a certain Extension of Intuitionistic Logic, *Dissertationes Mathematicae* 167, Institute of Mathematics, Polish Academy of Sciences, Warsaw 1980, 62 pp.

[31] G. Restall, Multiple Conclusions, in: P. Hájek, L. Valdes-Villanueva, and D. Westerstahl (eds.), *Logic Methodology and Philosophy of Science. Proceedings of the Twelfth International Congress*, King's College Publications, London, 2005, 189–205.

[32] R. Routley, Semantical Analyses of Propositional Systems of Fitch and Nelson, *Studia Logica*, 33 (1974), 283–298.

[33] P. Schroeder-Heister, Schluß und Umkehrschluß Ein Beitrag zur Definitionstheorie, to appear in: C.F. Gethmann (ed.), *Lebenswelt und Wissenschaft. Akten des XXI. Deutschen Kongresses für Philosophie 2009. Deutsches Jahrbuch Philosophie*, Vol. 3, 2010.

[34] Y. Shramko und H. Wansing, Some Useful 16-valued Logics: How a Computer Network Should Think, *Journal of Philosophical Logic* 34 (2005), 121–153.

[35] R. Thomason, A Semantical Study of Constructive Falsity, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 15 (1969), 247–257.

[36] I. Urbas, Dual-Intuitionistic Logic, *Notre Dame Journal of Formal Logic* 37 (1996), 440–451.

[37] D. Vakarelov, Notes on N-lattices and Constructive Logic with Strong Negation, *Studia Logica* 36 (1977), 109–125.

[38] D. Vakarelov, Constructive Negation on the Basis of Weaker Versions of Intuitionistic Negation, *Studia Logica* 80 (2005), 393–430.

[39] D. Vakarelov, Non-classical Negation in the Works of Helena Rasiowa and Their Impact on the Theory of Negation, *Studia Logica* (84), 2006, 105–127.

[40] H. Wansing, *The Logic of Information Structures*, Springer Lecture Notes in AI 681, Springer-Verlag, Berlin, 1993.

[41] H. Wansing, *Displaying Modal Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[42] H. Wansing, Negation, in: L. Goble (ed.), *The Blackwell Guide to Philosophical Logic*, Basil Blackwell Publishers, Cambridge/MA., 2001, 415–436.

[43] H. Wansing, Connexive Modal Logic, in: R. Schmidt et al. (eds.), *Advances in Modal Logic. Volume 5*, King's College Publications, London, 2005, 376–383 (http://www.aiml.net/volumes/volume5/).

[44] H. Wansing, On the Negation of Action Types: Constructive Concurrent PDL, in: P. Hájek, L. Valdes-Villanueva, and D. Westerstahl (eds.), *Logic Methodology and Philosophy of Science. Proceedings of the Twelfth International Congress*, King's College Publications, London, 2005, 207–225.

[45] H. Wansing, Connexive Logic, The Stanford Encyclopedia of Philosophy (Winter 2006 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2006/entries/logic-connexive/>.

[46] H. Wansing, Logical Connectives for Constructive Modal Logic, *Synthese* 150 (2006), 459–482.

[47] H. Wansing, A Note on Negation in Categorial Grammar, *Logic Journal of the IGPL* 15 (2007), 271–286.

[48] H. Wansing, Constructive Negation, Implication, and Co-implication, *Journal of Applied Non-Classical Logics* 18 (2008), 341-364.

[49] H. Wansing und Y. Shramko, Harmonious Many-valued Propositional Logics and the Logic of Computer Networks, in: C. Dégremont, L. Keiff and H. Rückert (ed.), *Dialogues, Logics and Other Strange Things. Essays in Honour of Shahid Rahman*, College Publications, London, 2008, 491-516.

[50] H. Wansing und Y. Shramko, Suszko's Thesis, Inferential Many-valuedness, and the Notion of a Logical System, *Studia Logica* 88 (2008), 405–429, 89 (2008), 147.

[51] T. Williamson, *Knowledge and Its Limits*, Oxford University Press, Oxford, 2000.

[52] F. Wolter, On Logics with Coimplication, *Journal of Philosophical Logic* 27 (1998), 353–387.