A Remark on a Peculiarity in the Functor Semantics for Superintuitionistic Predicate Logics with (or without) Equality

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Abstract

We notice the following slightly curious (and perhaps, slightly unexpected) logical property of the functor semantics for superintuitionistic predicate logics, contrasting with a well-known property of the usual Kripke semantics. Namely, for a category $\mathcal{C}$ its logic (i.e., the logic of all $\mathcal{C}$-sets with the given, fixed $\mathcal{C}$) in general is not reducible to cones (i.e., restrictions of $\mathcal{C}$ to upward closed rooted subsets of its frame representation $W = Ob(\mathcal{C})$). Related notions and observations are discussed as well.

Keywords: superintuitionistic predicate logics, Kripke semantics, functor semantics, categories.

Recall that a *cone* (a point-generated subframe) in an intuitionistic propositional Kripke frame (i.e., a pre-ordered, or in particular, a partially ordered set) W is $W^u = \{v \in W \mid u \leq v\}$ (for $u \in W$), ordered by the restriction of $\leq$ from W to W^u. It is well known that the (propositional) validity in Kripke frames is reducible to that in their cones, i.e.,

$$\mathbf{PL}(W) = \bigcap_{u \in W} \mathbf{PL}(W^u), \tag{0}$$

where $\mathbf{PL}(W)$ is the propositional logic of W (i.e., the set of formulas valid in W). The similar reducibility to cones holds for predicate logics as well, i.e.,

$$\mathbf{KL}(W) = \bigcap_{u \in W} \mathbf{KL}(W^u) \quad \text{for a pre-ordered set } W, \tag{1}$$

$\mathbf{KL}(W)$ being the set of predicate formulas valid in all predicate Kripke frames $F = (W, \overline{D})$ (with systems of expanding non-empty domains $\overline{D} = (D_u : u \in W)$) based on W; this claim holds for logics both without and with equality (in

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the latter case we will write $\mathbf{KL}^{=}(W)$). This statement is well known and usually is considered as obvious, however it is not so straightforward. Namely, an immediate predicate counterpart of (0) looks as follows:

$$\mathbf{L}(F) = \bigcap_{u\in W} \mathbf{L}(F^u) \quad \text{for a predicate Kripke frame } F \text{ based on } W, \qquad (1)'$$

where the cone F^u (for $u \in W$) is the restriction of F to W^u. Now, to deduce (1), the following simple claim needs to be used:

$$\begin{array}{l}\text{every frame } F_0 \text{ over } W^u \text{ can be extended} \\ \qquad\qquad \text{to a frame } F \text{ over } W \text{ such that } F^u = F_0.\end{array} \qquad (1)''$$

This auxiliary claim for the ordinary predicate Kripke semantics is rather obvious. Nevertheless, for more general Kripke-style semantics it becomes not so trivial. Here we consider the functor semantics [2] for superintuitionistic predicate logics and describe a natural counterpart of $(1)''$ that fails for this semantics; moreover, we show that the corresponding counterpart of (1) fails for predicate logics with equality. We believe that it would fail for logics without equality as well, although we do not have such an example. This observation confirms that the behavior of rather general predicate Kripke-style semantics often differs from that of the propositional Kripke semantics.

1 Preliminary notions: The functor semantics

We consider *predicate formulas* (without or with equality, and in any case without function symbols) built as usual, by using the connectives $\&, \vee, \to$, the propositional constant $\bot$ ('falsity'), and the quantifiers $\forall, \exists$. We use the standard abbreviations: $(A \leftrightarrow B) = (A \to B)\&(B \to A)$ and $\neg A = (A \to \bot)$.

We regard *superintuitionistic predicate logics* (without and with equality) in the usual way, i.e., as sets of predicate formulas containing all axioms of intuitionistic (Heyting) predicate logic $\mathbf{QH}$ (or $\mathbf{QH}^{=}$ for the case with equality), and closed under modus ponens, generalization, and substitution of arbitrary formulas for atomic ones.

To begin with, let us recall necessary notions related to the functor semantics, see [2] (also cf. [1, Sect. 5.6: Def. 5.6.3 etc.] or e.g. [6, Sect.4.1]).

Let $\mathcal{C}$ be a category with a frame representation W. This means that $W = Ob(\mathcal{C})$ is the set of objects of $\mathcal{C}$ pre-ordered by the following relation:

$u \leq v$ iff $\mathcal{C}(u,v) \neq \varnothing$, i.e., iff in $\mathcal{C}$ there exists a morphism from u to v.

A $\mathcal{C}$-*set* (a SET-valued functor or a presheaf over $\mathcal{C}$, inhabited, i.e., with non-emptiness assumption) is a triple $\mathbb{F} = (W, \overline{D}, \overline{E})$ in which $\overline{D} = (D_u : u \in W)$ is a family of disjoint non-empty domains and $\overline{E} = (E_\mu : \mu \in Mor(\mathcal{C}))$ is a family of functions with $E_\mu : D_u \to D_v$ whenever $\mu \in \mathcal{C}(u,v)$ (i.e., μ is a morphism from u to v). As usual, it is required that $E_{\mu\circ\mu'} = E_{\mu'} \circ E_\mu$ for $\mu \in \mathcal{C}(u,v)$, $\mu' \in \mathcal{C}(v,w)$ (i.e., $E_{\mu\circ\mu'}(a) = E_{\mu'}(E_\mu(a))$ for any $a \in D_u$), and $E_{1_u} = 1_{D_u}$ (the identity function on D_u corresponds to the identity morphism $1_u \in \mathcal{C}(u,u)$, $u \in W$). Sometimes we can admit $\mathcal{C}$-sets with non-disjoint domains; in this case we regard them as $\mathcal{C}$-sets with disjoint domains $D'_u = \{u\} \times D_u = \{\langle u, a\rangle \mid a \in D_u\}$.

Let $D_n = \bigcup_{u\in W} (D_u)^n$ for $n>0$ and $D_0 = W$. We define a pre-order $\leq_n$ on D_n (for $n>0$) by:

$$[\,(a_1,\dots,a_n) \leq_n (b_1,\dots,b_n)\,] \text{ iff } \exists\mu\in Mor(\mathcal{C})\,[\,\bigwedge_{i=1}^{n}(E_\mu(a_i) = b_i)\,]\,;$$

sometimes we write $(E_\mu(\mathbf{a})=\mathbf{b})$ for $\bigwedge_{i=1}^{n}(E_\mu(a_i)=b_i)$, where $\mathbf{a}=(a_1,\dots,a_n)$ and $\mathbf{b}=(b_1,\dots,b_n)$.[2] We identify $\leq_0$ with the original pre-order $\leq$ on $D_0=W$.

Finally let: $[\,(a_1,\dots,a_n) <_n (b_1,\dots,b_n)\,]$ iff
$[\,(a_1,\dots,a_n) \leq_n (b_1,\dots,b_n)\,]\,\&\,\neg[\,(b_1,\dots,b_n) \leq_n (a_1,\dots,a_n)\,]$.

A *valuation* in a $\mathcal{C}$-set $\mathbb{F}$ is a function ξ sending every n-place predicate symbol P to an upward closed (by $\leq_n$) subset $\xi(P)$ of D_n.

A valuation ξ in $\mathbb{F}$ gives rise to the forcing relation $u \vDash_\xi A(\mathbf{a})$ between points $u\in W$ and formulas $A(\mathbf{a})$ (with parameters replaced by elements of D_u); here $\mathbf{a}=(a_1,\dots,a_n)\in(D_u)^n$. Namely, $\vDash$ is defined inductively:[3]

$u \vDash P(\mathbf{a}) \Leftrightarrow (\mathbf{a} \in \xi(P))$ for a predicate symbol P; $\quad u \nvDash \bot$;
$u \vDash (B\&C)(\mathbf{a}) \Leftrightarrow (u \vDash B(\mathbf{a}))\,\&\,(u \vDash C(\mathbf{a}))$;
$u \vDash (B \vee C)(\mathbf{a}) \Leftrightarrow (u \vDash B(\mathbf{a})) \vee (u \vDash C(\mathbf{a}))$;
$u \vDash (B \to C)(\mathbf{a}) \Leftrightarrow \forall v \geq u\, \forall \mathbf{b}\in(D_v)^n\, [(\mathbf{a} \leq_n \mathbf{b})\&(v \vDash B(\mathbf{b})) \Rightarrow (v \vDash C(\mathbf{b}))]$;
$u \vDash \forall x B(\mathbf{a},x) \Leftrightarrow \forall v \geq u\, \forall \mathbf{b}\in(D_v)^n\, \forall c\in D_v\, [(\mathbf{a} \leq_n \mathbf{b}) \Rightarrow (v \vDash B(\mathbf{b},c))]$;
$u \vDash \exists x B(\mathbf{a},x) \Leftrightarrow \exists c\in D_u\, [u \vDash B(\mathbf{a},c)]$, and $u \vDash (a=b) \Leftrightarrow (a = b)$.

It is easily shown (by induction on A) that the forcing is preserved upwards:

$$(\mathbf{a} \leq_n \mathbf{b}) \;\Rightarrow\; (u \vDash A(\mathbf{a}) \Rightarrow v \vDash A(\mathbf{b}))$$

for any formula A and $\mathbf{a}\in(D_u)^n$, $\mathbf{b}\in(D_v)^n$, $u\leq v$.

A predicate formula A with parameters $\mathbf{x} = (x_1,\dots,x_n)$ is *true* w.r.t. ξ if $u \vDash A(\mathbf{a})$ for every $u \in W$ and $\mathbf{a} \in (D_u)^n$. A formula A is *valid* in a $\mathcal{C}$-set $\mathbb{F}$ (notation: $\mathbb{F} \vDash A$) if it is true w.r.t. all valuations in $\mathbb{F}$. The *predicate logic* (without or with equality) *of a $\mathcal{C}$-set* $\mathbb{F}$ is the set

$$\mathbf{L}^{(=)}(\mathbb{F}) = \{A \mid \text{ all substitution instances of } A \text{ are valid in } \mathbb{F}\};$$

again we write $\mathbf{L}(\mathbb{F})$ or $\mathbf{L}^{=}(\mathbb{F})$ for the logics without or with equality (respectively), and write $\mathbf{L}^{(=)}(\mathbb{F})$, when we mean both of these logics. It is known (cf. e.g. [1, Proposition 5.6.27] that $\mathbf{L}^{(=)}(\mathbb{F})$ is indeed a superintuitionistic logic.[4]

A formula is called *valid* in a category $\mathcal{C}$ (or *$\mathcal{C}$-valid*, for short) if it is valid in all $\mathcal{C}$-sets.

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[2] Here $(D_n,\leq_n)$ are well-defined, since domains D_u are disjoint. This is perhaps the only reason, why the disjointness of D_u is convenient and useful.

[3] Usually we omit subscript ξ (for readability) and write $\vDash$ for $\vDash_\xi$, $\vDash'$ for $\vDash_{\xi'}$, etc. Sometimes we may call $\vDash$ a valuation in $\mathbb{F}$ as well.

[4] By the way, there exists another, slightly more explicit description of the logic $\mathbf{L}^{(=)}(\mathbb{F})$; we present (and use) this description in Appendix (see Section 5).

Respectively, for a category $\mathcal{C}$ its *predicate logic* (without or with equality)

$$\mathbf{L}^{(=)}(\mathcal{C}) = \bigcap(\,\mathbf{L}^{(=)}(\mathbb{F}) : \mathbb{F} \text{ is a } \mathcal{C}\text{-set }\,)$$

is the set of formulas, all substitution instances of which are $\mathcal{C}$-valid.

Note that a usual predicate Kripke frame $F = (W, \overline{D})$ can be presented as a $\mathcal{C}$-set (based on W) with expanding domains (i.e., $D_u \subseteq D_v$ for $u \leq v$) and with inclusion mappings E_μ for $\mu \in \mathcal{C}(u, v)$.

2 The logics of categories: Non-reducibility to cones

Let $\mathcal{C}$ be a category based on a pre-ordered set W, and let $\mathbb{F}$ be a $\mathcal{C}$-set. Their *cones* $\mathcal{C}^u$ and $\mathbb{F}^u$ (for $u \in W$) are defined in a natural way, as the restrictions to the cone $W^u = \{v \in W \mid u \leq v\}$ of W; clearly, $\mathbb{F}^u$ is a $\mathcal{C}^u$-set. The restriction of a valuation in a $\mathcal{C}$-set $\mathbb{F}$ to $\mathbb{F}^u$ is a valuation in $\mathbb{F}^u$ (for atomic formulas), and the corresponding forcing relation $\vDash$ in $\mathbb{F}^u$ (for all formulas) is obtained as the restriction of $\vDash$ from $\mathbb{F}$. On the other hand, any valuation in $\mathbb{F}^u$ can be easily extended to a valuation in $\mathbb{F}$. Hence we conclude that:

$$\mathbf{L}^{(=)}(\mathbb{F}) \;=\; \bigcap_{u \in W} \mathbf{L}^{(=)}(\mathbb{F}^u) \text{ for any } \mathcal{C}\text{-set } \mathbb{F} \text{ based on } W, \qquad (2)'$$

i.e., the natural counterpart of $(1)'$ for the functor semantics holds. On the other hand, a similar property for categories fails:

Theorem *There exists a category* $\mathcal{C}_0$ *over a three-element chain* $W_0 = \{v_0, v_1, v_2\}$ *(where* $v_0 < v_1 < v_2$*) such that*

$$\mathbf{L}^{=}(\mathcal{C}) \not\subseteq \mathbf{L}^{=}(\mathcal{C}^u) \qquad (\overline{2})$$

for $\mathcal{C} = \mathcal{C}_0$ *and* $u = v_1$.

Namely, define the category $\mathcal{C}_0$ (based on W_0) with the following morphisms: $\mathcal{C}_0(v, v) = \{1_v\}$ for all $v \in W_0$, $\mathcal{C}_0(v_0, v_1) = \{\mu_0\}$, $\mathcal{C}_0(v_1, v_2) = \{\mu_1, \mu_2\}$, $\mathcal{C}_0(v_0, v_2) = \{\mu^*\}$, and with the composition $\mu_0 \circ \mu_i = \mu^*$ for $i = 1, 2$.

Put $\Phi_0 = (\Phi_1 \to \Phi_2)$, where $\Phi_1 = \forall x \forall y\,[(x = y) \vee P \vee \neg P]$, $\Phi_2 = \neg P \vee \neg\neg P$.

Define the $\mathcal{C}_0^{v_1}$-set $\mathbb{F}_0 = (W_0^{v_1}, \overline{D}, \overline{E})$, where $D_{v_1} = \{a_0\}$, $D_{v_2} = \{a_1, a_2\}$, and $E_{\mu_i}(a_0) = a_i$ for $i = 1, 2$. Our theorem follows from the subsequent claim:

Lemma 1 *(1)* $\Phi_0 \notin \mathbf{L}^{=}(\mathbb{F}_0)$, *and so* $\Phi_0 \notin \mathbf{L}^{=}(\mathcal{C}_0^{v_1})$; *(2)* $\Phi_0 \in \mathbf{L}^{=}(\mathcal{C}_0)$.

Proof. (1) The substitution instance $\Phi_0^1(z) = (\Phi_1^1(z) \to \Phi_2^1(z))$, where $\Phi_1^1(z) = \forall x \forall y\,[(x = y) \vee P_1(z) \vee \neg P_1(z)]$, $\Phi_2^1(z) = \neg P_1(z) \vee \neg\neg P_1(z)$, is not valid in $\mathbb{F}_0$. Indeed, consider a valuation

$$v_j \vDash P_1(a_i) \;\Leftrightarrow\; (i = j = 2)$$

in $\mathbb{F}_0$. Then $v_1 \vDash \Phi_1^1(a_0)$ (since $v_2 \vDash P_1(a_i) \vee \neg P_1(a_i)$ for $i = 1, 2$, and D_{v_1} is one-element) and $v_1 \not\vDash \Phi_2^1(a_0)$ (since $v_2 \vDash P_1(a_2)$ and $v_2 \vDash \neg P_1(a_1)$).

(2) Let a substitution instance of Φ_0 be given: $\Phi_0^A(\mathbf{z}) = (\Phi_1^A(\mathbf{z}) \to \Phi_2^A(\mathbf{z}))$, where $\Phi_1^A(\mathbf{z}) = \forall x \forall y\,[(x=y) \vee A(\mathbf{z}) \vee \neg A(\mathbf{z})]$, $\Phi_2^A(\mathbf{z}) = \neg A(\mathbf{z}) \vee \neg\neg A(\mathbf{z})$, $\mathbf{z} = (z_1, \dots, z_n)$ being the list of parameters of A (all z_i are distinct from x, y). Let $\mathbb{F} = (W_0, \overline{D}, \overline{E})$ be a $\mathcal{C}_0$-set. Suppose that $u \vDash \Phi_1^A(\mathbf{d})$ and $u \nvDash \Phi_2^A(\mathbf{d})$ for some $u \in W_0$ and $\mathbf{d} \in (D_u)^n$. Then $v_2 \vDash A(\mathbf{d}')$ and $v_2 \vDash \neg A(\mathbf{d}'')$ for some $\mathbf{d}', \mathbf{d}'' \in (D_{v_2})^n$ such that $\mathbf{d} <_n \mathbf{d}'$ and $\mathbf{d} <_n \mathbf{d}''$ (and so $u < v_2$). Then clearly, $u \nvDash A(\mathbf{d})$ and $u \nvDash \neg A(\mathbf{d})$, hence the domain D_u is one-element (since $u \vDash (a_1 = a_2)$ for any $a_1, a_2 \in D_u$).

First, if $u = v_0$, then $\mathbf{d}' = E_{\mu^*}(\mathbf{d}) = \mathbf{d}''$, and this leads to a contradiction.

Second, let $u = v_1$, $D_u = \{a_1\}$. Then $\mathbf{d} = (a_1, \dots, a_1) = E_{\mu_0}(\mathbf{d}^*)$, where $\mathbf{d}^* = (a_0, \dots, a_0)$ for an arbitrary $a_0 \in D_{v_0}$. Now, $\mathbf{d}' = E_{\mu_i}(\mathbf{d})$ for some $i \in \{1, 2\}$, hence $\mathbf{d}' = E_{\mu_i}(E_{\mu_0}(\mathbf{d}^*)) = E_{\mu^*}(\mathbf{d}^*)$, and similarly $\mathbf{d}'' = E_{\mu^*}(\mathbf{d}^*)$. Thus $\mathbf{d}' = \mathbf{d}''$ again. □

The key idea. For our category $\mathcal{C} = \mathcal{C}_0$ *and* $u = v_1$ we have:

$$\text{there exists a } \mathcal{C}^u\text{-set } \mathbb{F}_0 \text{ that cannot be extended to a } \mathcal{C}\text{-set } \mathbb{F} \text{ such that } \mathbb{F}^u = \mathbb{F}_0. \qquad (\overline{2})''$$

Clearly, this property $(\overline{2})''$ holds for any category $\mathcal{C}$ (based on W) and $u \in W$ satisfying $(\overline{2})$.

We conjecture that the peculiarity $(\overline{2})$ actually transfers to logics without equality as well, i.e., we hope that there exists a category $\mathcal{C}$ (based on W) such that

$$\mathbf{L}(\mathcal{C}) \not\subseteq \mathbf{L}(\mathcal{C}^u) \text{ for some } u \in W. \qquad (\overline{2})$$

However we could not construct a formula without equality that 'reflects' and exploits the property of our category $\mathcal{C}_0$, which does not allow to extend the $\mathcal{C}_0^{v_1}$-set $\mathbb{F}_0$ to a $\mathcal{C}_0$-set.

On the other hand, the following claim is obvious:

Lemma 2 *Let $\mathcal{C}$ be a category based on W. A formula A is $\mathcal{C}$-valid if it is $\mathcal{C}^u$-valid for all $u \in W$.*

Indeed, if a $\mathcal{C}$-set $\mathbb{F} \nvDash A$ is given, then $u \nvDash A$ for some $u \in W$ and a valuation $\vDash$ in $\mathbb{F}$. Hence $u \nvDash^u A$ for the restriction $\vDash^u$ of $\vDash$ to $\mathbb{F}^u$, and so $\mathbb{F}^u \nvDash A$. □

Hence we obtain

Corollary 3 *For any category $\mathcal{C}$ based on W:*

$$\bigcap_{u \in W} \mathbf{L}^{(=)}(\mathcal{C}^u) \subseteq \mathbf{L}^{(=)}(\mathcal{C}). \qquad (2)$$

Clearly, this inclusion is proper iff $\mathcal{C}$ satisfies $(\overline{2})$ (for the logics without or with equality, respectively).

3 Reducibility to cones for ∀-positive formulas

Now we describe a class of formulas, for which $\mathcal{C}$-validity is reducible to $\mathcal{C}^u$-validity (for all categories $\mathcal{C}$).

Recall that an occurrence of a subformula or a quantifier etc. in a formula A is called *positive* (or *negative*) if it occurs in an even (respectively, odd) number of premises of implications. We call a formula A (without or with equality) *∀-positive* if all occurrences of ∀ in A are positive and all occurrences of ∃ in A are negative. Similarly, a formula A is *∃-positive* if all occurrences of ∃ in A are positive and all occurrences of ∀ in A are negative. The following simple lemma gives an inductive description of these notions:

Lemma 4

(1) *Any atomic formula is both ∀-positive and ∃-positive.*

(2) *Formulas $A_1 \& A_2$, $A_1 \vee A_2$ are ∀-positive (or ∃-positive) iff both A_1 and A_2 are ∀-positive (respectively, ∃-positive).*

(3) *A formula $(A_1 \to A_2)$ is ∀-positive iff*
A_1 is ∃-positive and A_2 is ∀-positive.
A formula $(A_1 \to A_2)$ is ∃-positive iff
A_1 is ∀-positive and A_2 is ∃-positive.

(4) *A formula $\forall x\, A$ is ∀-positive iff A is ∀-positive.*
Also $\forall x\, A$ is not ∃-positive (for any A).

(5) *A formula $\exists x\, A$ is ∃-positive iff A is ∃-positive.*
Also $\exists x\, A$ is not ∀-positive (for any A).

We present a proof of the subsequent claim in Appendix (see Section 5):

Proposition 1 *Let $\mathcal{C}$ be a category based on W, and let A be a ∀-positive formula (without or with equality). Then:*

$$A \in \mathbf{L}^{(=)}(\mathcal{C}) \;\Rightarrow\; A \in \mathbf{L}^{(=)}(\mathcal{C}^u) \text{ for all } u \in W.$$

Therefore,

$$(A \in \mathbf{L}^{(=)}(\mathcal{C})\,) \text{ iff } (A \in \mathbf{L}^{(=)}(\mathcal{C}^u) \text{ for all } u \in W),$$

for any ∀-positive formula A (since the converse implication readily follows from Corollary 3).

Let us call all negative occurrences of ∀ and all positive occurrences of ∃ in a formula A its *critical* occurrences. Clearly, a formula is ∀-positive iff it has no critical occurrences of quantifiers.

The formula Φ_0 constructed in Section 2 (for our Theorem) is intuitionistically equivalent to the formula

$$\Phi_0' = (\Phi_1' \to \Phi_2), \qquad \text{where } \Phi_1' = \exists x \forall y\, [(x = y) \vee P \vee \neg P],$$

since $\mathbf{QH}^{=} \vdash (\Phi_1 \leftrightarrow \Phi_1')$. The formula Φ_0' has only one critical occurrence $\forall y$ (and one non-critical $\exists x$). Also Φ_0' is equivalent to $\forall x\, \Phi_0''$, where

$\Phi_0'' = (\,\forall y\,[(x{=}y) \lor P \lor \neg P] \;\to\; \neg P \lor \neg\neg P\,)$.
This $\exists$-free formula has one critical occurrence $\forall y$ (and one non-critical $\forall x$). Moreover, $\forall x\,\Phi_0''$ can be replaced with the non-closed $\exists$-free formula $\Phi_0''(x)$ with only one occurrence of $\forall$ (clearly, critical). On the other hand, we do not know, whether Proposition 1 can be transferred to $\forall$-free formulas (with critical, i.e., positive occurrences of $\exists$).

4 The logics of pre-ordered sets: Reducibility to cones

In Section 2 we presented a natural counterpart of the claim (1) that fails for the functor semantics (at least, in the case with equality). Nevertheless, a more literal, straightforward counterpart of (1) holds.

Namely, let us define the *predicate logic* (without or with equality) *of* a pre-ordered set W *in the functor semantics* as follows:

$$\mathbf{FL}^{(=)}(W) = \bigcap (\,\mathbf{L}^{(=)}(\mathcal{C}) : \mathcal{C} \text{ is a category based on } W\,).$$

The following claim holds (the proof is given in Appendix, see Section 5):

Proposition 2 $\mathbf{FL}^{(=)}(W) \subseteq \mathbf{FL}^{(=)}(W^u)$ *for any pre-ordered* W *and* $u{\in}W$.

Therefore,

$$\mathbf{FL}^{(=)}(W) \;=\; \bigcap_{u\in W} \mathbf{FL}^{(=)}(W^u) \quad \text{for any pre-ordered set } W; \tag{3}$$

indeed, the converse inclusion readily follows from Corollary 3.

However, for logics without equality this claim is not interesting, because a strictly stronger statement actually holds (its proof requires extra preparation, techniques, and accuracy, and will be given in the continuation [5] of this paper):

Proposition 2^* $\mathbf{FL}(W) \;=\; \mathbf{QH}$ *(intuitionistic predicate logic) for a one-element partially ordered set* W, *and hence for any pre-ordered* W.

In other words, the logic without equality $\mathbf{FL}(W)$ does not depend on W.

So to say, the functor semantics is too powerful to be considered at the level of propositional Kripke bases (i.e., pre-ordered sets), because at this level it becomes degenerated (at least, for logics without equality). That is why we assume that the level of categories (i.e., the logics $\mathbf{L}^{(=)}(\mathcal{C})$ introduced in Section 1) is more adequate. In particular, (2) seems to be a more natural counterpart of (1) for the functor semantics than (3), and the peculiarity stated in Section 2 shows the limits of this correspondence between (1) and (2).

On the other hand, Proposition 2^* is not transferred to logics with equality $\mathbf{FL}^{=}(W)$. Namely, all these logics have the intuitionistic equality-free fragment, however they are distinct for different W. For example, it is easily seen that the formula

$$\Psi \;=\; [\;\forall x\,\forall y\,(x = y) \;\to\; P \lor \neg P\;]$$

belongs to $\mathbf{FL}^{=}(W)$ for a one-element W (since any $\mathcal{C}$-set over one-element partially ordered set with one-element domain is a classical one-element model), but does not belong, e.g., to $\mathbf{FL}^{=}(W)$ for a two-element chain. Actually,

$\Psi \in \mathbf{FL}^{=}(W)$ iff ($\leq$ is an equivalence relation on W)

(i.e., iff the skeleton of W is an antichain).

Moreover, for a propositional formula A, let us denote

$$\Psi^A = [\, \forall x \forall y\, (x = y) \rightarrow A \,].$$

Proposition 3 *For a propositional formula A and a pre-ordered set W we have:*

$\Psi^A \in \mathbf{FL}^{=}(W)$ iff $A \in \mathbf{PL}(W)$ *(the superintuitionistic propositional logic of W).*

Indeed, a $\mathcal{C}$-set with one-element domains is essentially a usual predicate Kripke frame with one-element constant domain; it is sufficient to identify its individual domains, i.e., we suppose that $D_u = D = \{a_0\}$ for all $u \in W$, and then any E_μ (for $\mu \in \mathcal{C}(u, v)$, $u \leq v$ in W) becomes the identical mapping on D.[5] □

This statement means that the propositional logic $\mathbf{PL}(W)$ is embeddable into the corresponding logic with equality $\mathbf{FL}^{=}(W)$. Therefore we conclude that:

Corollary 5 *There exists a continuum of different logics with equality of the form $\mathbf{FL}^{=}(W)$ (for different W).*

Indeed, there exists a continuum of Kripke-complete superintuitionistic propositional logics (cf. [3]). □

Open problem 1 Try to describe the logics with equality $\mathbf{FL}^{=}(W)$ for natural and simple partially ordered (or pre-ordered) sets W; e.g., for a one-element W.

We do not know, whether these logics are recursively (or finitely) axiomatizable; clearly, this question arises only for sets W, whose propositional logics $\mathbf{PL}(W)$ are recursively axiomatizable (e.g., for finite W). We do not know, whether $\mathbf{FL}^{=}(W)$ for a pre-ordered set W equals the logic of its partially ordered skeleton. And more questions remain open.

We say that a pre-ordered set W is $\mathbf{QH}^{=}$*-complete in the functor semantics* if $\mathbf{FL}^{=}(W) = \mathbf{QH}^{=}$. Proposition 3 implies the following simple consequence:

Corollary 6 *A pre-ordered set W is $\mathbf{QH}^{=}$-complete in the functor semantics only if $\mathbf{PL}(W) = \mathbf{H}$ (intuitionistic propositional logic).*

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[5] By the way, note that here we mention only propositional formulas A, because it is easily seen that in a Kripke frame with one-element constant domain any predicate formula A is reducible to a propositional formula $\alpha(A)$, cf. [4, Sect. 5 (and 6)].

Indeed, definitely $\mathbf{QH}^= \not\vdash \Psi^A$ for an intuitionistically unprovable propositional formula A. □

Now, let us say that W is $\mathbf{QH}^=$-*complete in the Kripke semantics* if $\mathbf{KL}^=(W) = \mathbf{QH}^=$. The most familiar examples are the infinite tree ω^* of all finite sequences of natural numbers or the binary tree $\{0,1\}^*$ of all finite $\{0,1\}$-sequences, etc.

Open problem 2 Does there exist a pre-ordered (or a partially ordered) set W that is $\mathbf{QH}^=$-complete in the functor semantics but not in the Kripke semantics?

Let us mention two possible candidates.

First, let W_{fin} be the disjoint union of all finite trees (or e.g., all Jaskowski's trees). Then $\mathbf{PL}(W_{fin}) = \mathbf{H}$, whereas $\mathbf{KL}^=(W) \neq \mathbf{QH}^=$, because e.g. the well-known Kuroda's formula is Kripke-valid in W_{fin}:

$$K = \neg\neg\forall x(P(x) \vee \neg P(x)).$$

On the other hand, $K \notin \mathbf{FL}^=(W_{fin})$, since it is easily shown that $K \notin \mathbf{FL}^=(W)$ for a one-element W (and then apply Proposition 2).

Second, let $W = \overline{\omega^*}$ be the tree obtained by adding maximal points above all points of ω^* (or above all branches in ω^*, etc.), or a similar binary tree $W = \overline{\{0,1\}^*}$. Then $\mathbf{KL}^=(W) = (\mathbf{QH}^= + K)$. By the way, note that it is easily shown that $\mathbf{FL}^=(\overline{\omega^*}) \subseteq \mathbf{FL}^=(W_{fin})$, because there exist p-morphisms of $\overline{\omega^*}$ onto all finite trees. So $\overline{\omega^*}$ would be definitely $\mathbf{QH}^=$-complete if W_{fin} were.

5 Appendix: The proofs of Propositions 1 and 2

To establish Proposition 1, we use the following notion.

Let $\mathcal{C}$ be a category based on W, and let $\mathbb{F}', \mathbb{F}''$ be $\mathcal{C}$-sets. Say that $\mathbb{F}'$ is a $\mathcal{C}$-*subset* of $\mathbb{F}''$ (and $\mathbb{F}''$ is a $\mathcal{C}$-*extension* of $\mathbb{F}'$) if $D'_u \subseteq D''_u$ for every $u \in W$ and E'_μ is the restriction of E''_μ to D'_u for $\mu \in \mathcal{C}(u,v)$, $u \le v$; naturally, we suppose that the functions E'_μ are well-defined, i.e., $E''_\mu(D'_u) \subseteq D'_v$ for any $\mu \in \mathcal{C}(u,v)$. Now, a valuation ξ'' in $\mathbb{F}''$ gives rise to the valuation ξ' in $\mathbb{F}'$ defined as the restriction of ξ'' to $\mathbb{F}'$ (for atomic formulas); note that the corresponding forcing relation $\models'$ in $\mathbb{F}'$ in general is not the restriction of $\models''$ from $\mathbb{F}''$ to $\mathbb{F}'$ (for non-atomic formulas).

Lemma 7 *Let $\mathcal{C}$ be a category based on W, let $\mathbb{F}'$ be a $\mathcal{C}$-subset of $\mathbb{F}''$, and let ξ' be the restriction (to $\mathbb{F}'$) of a valuation ξ'' in $\mathbb{F}''$. Let $u \in W$, and let $A(\mathbf{a})$ be a formula (with parameters replaced by elements of D'_u). Then:*

(1) *if A is $\forall$-positive, then: $(u \models'' A(\mathbf{a})$ in $\mathbb{F}'') \Rightarrow (u \models' A(\mathbf{a})$ in $\mathbb{F}')$;*

(2) *if A is $\exists$-positive, then: $(u \models' A(\mathbf{a})$ in $\mathbb{F}') \Rightarrow (u \models'' A(\mathbf{a})$ in $\mathbb{F}'')$.*

Proof is obtained by a straightforward induction on A (use Lemma 4 !). Let us mention the three principal cases (other cases are obvious).

(I) $A = (A_1 \to A_2)$.

(1) Let A be $\forall$-positive, so A_1 is $\exists$-positive and A_2 is $\forall$-positive. Assume that $u \not\models' A(\mathbf{a})$, i.e., $v \models' A_1(E'_\mu(\mathbf{a}))$ and $v \not\models' A_2(E'_\mu(\mathbf{a}))$ for some $v \geq u$, $\mu \in \mathcal{C}(u,v)$. Then $v \models'' A_1(E''_\mu(\mathbf{a}))$ and $v \not\models'' A_2(E''_\mu(\mathbf{a}))$ by inductive hypothesis (note that here $E''_\mu(\mathbf{a}) = E'_\mu(\mathbf{a})$), and so $u \not\models'' A(\mathbf{a})$. (2) is shown similarly.

(II) $A = \exists x\, A_0$.

(2) Let A (and so A_0) be $\exists$-positive. Assume that $u \models' A(\mathbf{a})$, i.e., $u \models' A_0(b, \mathbf{a})$ for some $b \in D'_v \subseteq D''_v$. Then $u \models'' A_0(b, \mathbf{a})$, and so $u \models'' A(\mathbf{a})$.

Also, (1) is vacuous, since A definitely is not $\forall$-positive.

(III) $A = \forall x\, A_0$.

(1) Let A (and so A_0) be $\forall$-positive. Now we have to assume that $u \not\models' A(\mathbf{a})$, i.e., $v \not\models' A_0(b, E'_\mu(\mathbf{a}))$ for some $v \geq u$, $\mu \in \mathcal{C}(u,v)$, $b \in D'_v \subseteq D''_v$. Then we conclude that $v \not\models'' A_0(b, E''_\mu(\mathbf{a}))$, and so $u \not\models'' A(\mathbf{a})$.

And (2) is vacuous again. □

Any $\mathcal{C}$-set $\mathbb{F}$ gives rise to its *simple extension* $\mathbb{F}^+$, where $D^+_u = D_u \cup \{e_u\}$ (here $e_u \notin D_u$) for all $u \in W$, and $E^+_\mu\,|_{D_u} = E_\mu$, $E^+_\mu(e_u) = e_v$ for $\mu \in \mathcal{C}(u,v)$.

Lemma 8 *Let $\mathcal{C}$ be a category based on W, let $u_0 \in W$, and let A be a $\forall$-positive formula. Then the $\mathcal{C}$-validity of A implies its $\mathcal{C}^{u_0}$-validity.*

Proof. Let a $\forall$-positive A be not $\mathcal{C}^{u_0}$-valid., i.e., $w \not\models_0 A(\mathbf{a})$ for some valuation ξ_0 in a $\mathcal{C}^{u_0}$-set $\mathbb{F}_0$, some $w \in W^{u_0}$, and $\mathbf{a} \in (D_w)^n$. Consider a $\mathcal{C}^{u_0}$-set $\mathbb{F}$ such that $\mathbb{F}^{u_0} = (\mathbb{F}_0)^+$, $D_u = \{e_u\}$ for $u \notin W^{u_0}$, and $E^+_\mu(e_u) = e_v$ for $\mu \in \mathcal{C}(u,v)$ (for <u>all</u> $u, v \in W$, $u \leq v$). Extend ξ_0 to a valuation ξ in $\mathbb{F}$; e.g. put a valuation $\xi(P) = \xi_0(P)$ for all predicate symbols P (so atoms are true <u>only</u> at points from W^{u_0}). Then $w \not\models A(\mathbf{a})$ by Lemma 7(1), and so A is not $\mathcal{C}$-valid. □

However, this claim does not immediately imply Proposition 1, since a substitution instance of a $\forall$-positive formula in general is not $\forall$-positive. So we apply the following, slightly more explicit description of the logic $\mathbf{L}^{(=)}(\mathbb{F})$ of a $\mathcal{C}$-set.

Let A be a predicate formula, and let k_i-ary symbols P_i (for $i = 1, \ldots, m$) be all predicate symbols occurring in A (besides equality!). Now, for $n \geq 0$, let $\mathbf{y} = (y_1, \ldots, y_n)$ be a list of different variables not occurring in A, and let Q_i be different $(k_i + n)$-ary predicate symbols. Then *n-shift* A^n of A is the substitution instance of A obtained by simultaneously replacing of all atomic subformulas $P_i(\mathbf{x})$ in A with $Q_i(\mathbf{x}, \mathbf{y})$ $(i = 1, \ldots, m)$.

It is known that $A \in \mathbf{L}^{(=)}(\mathbb{F})$ iff all A^n (for $n \geq 0$) are valid in $\mathbb{F}$ (see [1, Proposition 5.6.26(2)] or cf. e.g. [6, Sect.4.1]). So to say, to check, whether $A \in \mathbf{L}^{(=)}(\mathbb{F})$, it is sufficient to check the validity of A 'with an arbitrary number of additional parameters' (and then the validity for all substitution instances of A readily follows). Therefore, for a category $\mathcal{C}$ we conclude that

$$A \in \mathbf{L}^{(=)}(\mathcal{C}) \text{ iff all } A^n \text{ (for } n \geq 0) \text{ are } \mathcal{C}\text{-valid.}$$

Clearly, for a $\forall$-positive A, all A^n are $\forall$-positive as well, and therefore now Proposition 1 immediately follows from Lemma 8.

Next, Proposition 2 follows from $(2)'$ (see Section 2) and the subsequent:

Lemma 9 *Let W be a pre-ordered set and $u_0 \in W$. Then every category $\mathcal{C}$ based on W^{u_0} and a $\mathcal{C}$-set $\mathbb{F}$ can be extended to a category $\mathcal{C}^*$ based on W and a $\mathcal{C}^*$-set $\mathbb{F}^*$ such that $(\mathcal{C}^*)^{u_0} = \mathcal{C}$ and $(\mathbb{F}^*)^{u_0} = \mathbb{F}$.*

Proof. Put $\mathcal{C}^*(u,v) = \mathcal{C}(u,v)$ for $u, v \in W^{u_0}$, $\mathcal{C}^*(u,v) = \{1_{u,v}\}$ for $u, v \notin W^{u_0}$, $\mathcal{C}^*(u,v) = \{\langle \mu, u\rangle \mid \mu \in \mathcal{C}(u_0, v)\}$ for $u \notin W^{u_0}$, $v \in W^{u_0}$ (here $\langle \mu, u\rangle$ is a 'copy' of μ at $u \notin W^{u_0}$). Let us describe $\mu \circ \mu'$ for $\mu \in \mathcal{C}^*(u,v)$, $\mu' \in \mathcal{C}^*(v,w)$. First, $\mu \circ \mu'$ is taken from $\mathcal{C}$ for $u \in W^{u_0}$. Second, $1_{u,v} \circ 1_{v,w} = 1_{u,w}$ for $w \notin W^{u_0}$ and $1_{u,v} \circ \langle \mu, v\rangle = \langle \mu, u\rangle$ for $v \notin W^{u_0}$, $w \in W^{u_0}$ (here $\mu \in \mathcal{C}(u_0, w)$). Finally, $\langle \mu, u\rangle \circ \mu' = \langle \mu \circ \mu', u\rangle$ for $u \notin W^{u_0}$, $v \in W^{u_0}$ (here $\mu \in \mathcal{C}(u_0, v)$, $\mu' \in \mathcal{C}(v,w)$, so $\mu \circ \mu' \in \mathcal{C}(u_0, w)$). The associativity for the composition in $\mathcal{C}^*$ is easily checked.

Now we extend $\mathbb{F}$ to $\mathbb{F}^*$. Namely, put $D^*_u = D_u$ for $u \in W^{u_0}$ and $D^*_u = D_{u_0}$ for $u \notin W^{u_0}$. Also define E^*_μ for $\mu \in \mathcal{C}^*(u,v)$ as follows: $E^*_\mu = E_\mu$ for $u, v \in W^{u_0}$,

$E^*_{1_{u,v}} = 1_{D_{u_0}}$ (the identity function on $D_{u_0} = D^*_u = D^*_v$) for $u, v \notin W^{u_0}$,

$E^*_{\langle \mu, u\rangle} = E_\mu$ for $u \notin W^{u_0}$, $v \in W^{u_0}$ (here $\mu \in \mathcal{C}(u_0, v)$ and $D^*_u = D_{u_0}$).

The key property $E^*_{\mu \circ \mu'} = E^*_{\mu'} \circ E^*_\mu$ for $\mu \in \mathcal{C}^*(u,v)$, $\mu' \in \mathcal{C}^*(v,w)$ (cf. Section 1) obviously holds. □

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