Dynamic Mereotopology II: Axiomatizing some Whiteheadean Type Space-time Logics

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Abstract

In this paper we present an Whiteheadean style point-free theory of space and time. Here "point-free" means that neither space points, nor time moments are assumed as primitives. The algebraic formulation of the theory, called dynamic contact algebra (DCA), is a Boolean algebra whose elements symbolize dynamic regions changing in time. It has three spatio-temporal relations between dynamic regions: *space contact*, *time contact* and *preceding*. We prove a representation theorem for DCA-s of topological type, reflecting the dynamic nature of regions, which is a reason to call DCA-s *dynamic mereotopoly.* We also present several complete quantifier-free logics based on the language of DCA-s.

Keywords: Dynamic mereotopology, point-free theory of space and time, representation theorem, quantifier-free spatio-temporal logic.

Introduction

Alfred Whitehead is well-known as the co-author with Bertrand Russell of the famous book "Principia Mathematica". They intended to write a special part of the book related to the foundation of geometry, but due to some disagreements between them this part has not been written. Later on Whitehead formulated his own program for a new theory of space and time. The best articulation of this program is in this quote from [24], page 195 (boldface is ours):

> "...Our space concepts are concepts of relations between material things in space. Thus there is no such entity as a self-subsistent point. A point is merely the name for some peculiarity of the relations between matter which is, in common language, said to be in space.
>
> **It follows from relativity theory that a point should be definable in terms of the relations between material things.** So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning. ... **Similar explanations apply to time.** Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have

to be undertaken, and many ways of effecting these definitions will have to be tried and compared. **This is an unwritten chapter of mathematics**..."

According to the above program Whitehead should be considered as the initiator of point-free approach to the theory of space, known now as *Region Based Theory of Space (RBTS)*. In his famous book "Process and Reality" [26] Whitehead presented a more detailed program of how to build mathematical formalizations of some versions of RBTS. His main primitive is the notion of *region* as a formal analog of physical body, and some relations between regions like *part-of*, *overlap*, and *contact*. Whitehead shows how to define points, lines and planes by means of certain classes of regions, and in general how to rebuild the whole geometry on this new basis in an equivalent way. Other sources of RBTS are, for instance, de Laguna [5] and Tarski [21], who showed how to rebuild Euclidean geometry on the base of mereology and the primitive notion of *ball*. Survey papers about RBTS are [3,15,22].

Part-of and overlap are studied in *mereology* considered as a philosophical theory of parts and wholes [19], and according to Tarski (see [19]) mereology is equivalent in certain sense to Boolean algebra. Extensions of mereology with contact or some contact-like relations, is now called *mereotopology*, which can be considered as a theoretical tool for RBTS. Motivation for such a terminology is that the main point models of mereotopologies are topological. Recent forms of mereotopology are various notions of contact algebras (see [9,20,7]), which are Boolean algebras enriched with an additional relation called contact, and the simplest one for which we reserve the name "contact algebra", was introduced in [7]. Standard point models of contact algebras are the Boolean algebras of regular closed sets in topological spaces and two regular closed sets are in a contact if they have a nonempty intersection. The paper [7] contains point-free characterizations of contact algebras of regular closed sets of various classes of topological spaces. Let us mention that the pioneering work in this area was given by H. de Vries [6] but mainly oriented to applications in topology, which was one of the reasons his work to be unknown for a long time among the community of researchers interested in RBTS. Let us note also that Whitehead's ideas for RBTS were in a sense reinvented in computer science and Artificial Intelligence as a more suitable formalism for representing spatial information. This fact generated a very intensive study of spatial formalisms related to RBTS with various applications (see for this [4]).

Whitehead's theory of time was developed mainly in [25] and [26] and was called *Epochal Theory of Time (ETT)*. Whitehead claims that the theory of time should not be separated from the theory of space and their integrated theory has to be extracted from the existing things in reality and some of their spatio-temporal relations. This integrated theory should be point-free in a double sense: that both space points and time points (moments of time) should be definable by the other primitives of the theory. Unfortunately, unlike his program how to build mathematical theory of space, given in [26], Whitehead did not describe analogous program for his integrated theory of space and time. He presented his ideas for ETT quite informally and in a pure philosophical manner, which makes extremely difficult to extract from his texts clear mathematical theory corresponding to ETT. So, in this paper we will follow mainly Whitehead's ideas described in the quote cited above and will try to present an extension of RBTS, considered as an integrated theory of space and time, containing neither points, nor moments of time as primitive notions.

The present paper is a second one in the series of papers started by [23] with the aim to present some integrated theory of space and time in a point-free Whiteheadean style. The main idea in [23] was to obtain an analog of contact algebra, called *dynamic contact algebra* (DCA), which formalizes changing regions. The standard point model of DCA having explicit set T of time points (moments of time) is defined as follows. Suppose we are observing some area of regions. If they are not changing then their spatial structure forms a contact algebra. If the regions are changing in time then the information which we need for a given dynamic region a is to know its instance a_m at each moment of time m. So, a has to be considered as a vector, or a function defined in T, with coordinates a_m, $m \in T$. It is natural to consider a_m as an element of a contact algebra $(\mathbf{B}_m, C_m) = (B_m, 0_m, 1_m, \leq_m, ., +, *, C_m)$, (called coordinate contact algebra corresponding to the time moment m), assuming in this way that $\mathbf{B}_m$ as a snapshot of the whole state of affairs at the moment m. We may assume further that dynamic regions form a Boolean algebra with operations defined coordinatewise. Hence this Boolean algebra is a sublagebra of the Cartesian product of all coordinate algebras. Just to make things not very complicated we assumed in [23] that the set T has no internal structure of the intended time ordering and considered as primitives only two very simple spatio-temporal relations between dynamic regions: *stable contact* denoted by $C^{\forall}$, and *unstable contact* denoted by $C^{\exists}$, with the following definitions in the standard model:

$aC^{\forall}b$ iff $(\forall m \in T)a_m C_m b_m$,

i.e. a is in a *stable contact* with b if a and b are in a *contact* at each moment of time m. Analogously

$aC^{\exists}b$ iff $(\exists m \in T)a_m C_m b_m$,

i.e. a is in an *unstable contact* with b if a and b are in a *contact* at some moment of time m.

Note that these relations do not depend on any time ordering. In order to make all this free of the explicit use of time we axiomatized this structure in an abstract algebraic form obtaining an abstract definition of DCA. The main result in [23] was a representation theorem of DCA into standard dynamic contact algebras.

In the present paper we extend the point-free approach from [23] considering standard dynamic models in which the set of time moments T is supplied with an intended time ordering denoted by $\prec$ and satisfying some reasonable conditions. So, in order to introduce it we need a new relation between dynamic regions depending on time order. There are many such relations and the main problem is to find a suitable one which guarantees the expected representation theorem with some natural properties of the time order. In this paper we use one, denoted by $\mathcal{B}$, called *precedence relation*, with a very simple formal properties in the expected axiomatization. The intuitive meaning of $a\mathcal{B}b$ is "a exists at some moment of time and b exists at a later moment". In the standard model this definition sounds as follows:

- $a\mathcal{B}b$ iff $(\exists m, n \in T)(m \prec n$, and $a_m \neq 0_m$ and $b_n \neq 0_n)$.

Here $a_m \neq 0_m$ just says that a exists at the moment m, and the same for $b_n \neq 0_n$.

We consider also two other relations:

- $aC^s b$ iff $(\exists m \in T)a_m C_m b_m$, *space contact*,
- $aC^t b$ iff $(\exists m \in T)a_m \neq 0_m$ and $b_m \neq 0_m$, *time contact*.

Space contact coincides with the unstable contact $C^\exists$ and we renamed it, because it is the natural contact relation between dynamic regions which ensures that the coordinate algebras $\mathbf{B}_m$ are contact algebras and consequently it is responsible for the definition of space points. While space contact means having a common space point, time contact indeed means having a common time point and it is also responsible for the definition of time points. Note that it is a kind of *simultaneity* or *contemporaneity* relation mentioned in Whitehead texts. Let us note that Whitehead did not use something like our precedence relation $a\mathcal{B}b$. We take it by two reasons: first it is responsible in the definition of time ordering, and second, although its simplicity in the axiomatization, it together with the time contact is able to characterize point-free many natural properties of time ordering: left and right seriality, reflexivity, transitivity, linearity, density, up-directness and down-directness, different subsets of which define different space-time theories.

The rest of the paper is organized as follows. Section 1 lists some facts about contact and precontact algebras. In Section 2 we introduce dynamic models of space with explicit moments of time and time ordering. Section 3 is devoted to the main notion of the paper - dynamic contact algebra (DCA). In Section 4 we developed representation theory of DCA-s. Section 5 contains some quantifier-free constraint logics based on the language of DCA. In Section 6 we discuss relations with other works, some open problems, and plans for future research.

1 Preliminaries about contact and precontact algebra

Definition 1.1 ([7]) *Let $(B, 0, 1, ., +, ^*)$ be a non-degenerate Boolean algebra with * denoting its Boolean complement. A binary relation C in B is called a contact relation if it satisfies the following conditions:*

$(C1)$ *If aCb, then $a, b \neq 0$,*
$(C2)$ *$aC(b + c)$ iff aCb or aCc,*
$(C3)$ *If aCb, then bCa,*
$(C4)$ *If $a.b \neq 0$, then aCb.*

If B is a Boolean algebra and C a contact relation *in B then the pair (B, C) is called a* contact algebra *over B. The elements of B are called* regions*. The negation of C is denoted by $\overline{C}$. The element 0 is called zero region and is considered as a region which does not exist (here "exists" is considered as a predicate). Then $a \neq 0$ means that a exists. If $a \leq b$, where $\leq$ is the Boolean* *ordering, then this will be read as* a *is a part of* b*, so* $\leq$ *is the mereological relation part-of. The region* 1 *is called the unit region, the region which has as its parts all other regions. The relation* aOb *iff* $a.b \neq 0$ *is the mereological relation* overlap *in* B*.*

The following lemma lists some easy consequences of axioms.

Lemma 1.2 *(i)* aCb *and* $a \leq a'$ *and* $b \leq b'$ *implies* $a'Cb'$*,*
(ii) $a \neq 0$ *iff* aCa

Let us note that each Boolean algebra has at least two contact relations: the overlap O, which by axiom (C4) is the *smallest contact* in B, and $aC^{max}b \leftrightarrow_{def} a \neq 0$ and $b \neq 0$, which by axiom C1 is the *maximal contact* in B.

Contact algebras of regular closed sets in a topological space. Let X be a non-empty topological space with the closure and interior operations denoted respectively by $Cl(a)$ and $Int(a)$. A subset a of X is *regular closed* if $a = Cl(Int(a))$. The set of all regular closed subsets of X is denoted by $RC(X)$. It is a well-known fact that regular closed sets with the operations $a+b = a \cup b$, $a.b = Cl(Int(a \cap b))$, $a^* = Cl(X \setminus a)$, $0 = \varnothing$ and $1 = X$ form a Boolean algebra. If we define the contact by $a\, C_X\, b$ iff $a \cap b \neq \varnothing$ then C_X satisfies the axioms $(C1)$–$(C4)$. Such a contact is called standard contact for regular closed sets and the contact algebra of this example and any its subalgebra is said to be *standard contact algebra of regular closed sets*, or simply *topological contact algebra*. Topological space with a closed base of regular closed sets is called *semiregular* (for other topological notions related to contact algebras see [7] or [22]).

The following representation theorem for contact algebras is proved in [7]:

Theorem 1.3 (Topological representation thm. for contact algebras) *For every contact algebra* (B, C) *there exists a semi-regular and compact* T_0 *space* X *and an embedding* h *into the contact algebra* $RC(X)$*.*

Theorem 1.4 ([23] Joint embedding theorem for contact algebras) *Let* (B_t, C_t)*,* $t \in T$ *be a nonempty family of contact algebras. Then there exist a semiregular and compact* T_0 *space* X *and a family of embeddings* g_t *of* (B_t, C_t) *into* $RC(X)$*,* $t \in T$*.*

Abstract points of contact algebras. The abstract points which are used in the representation theory of contact algebras developed in [7] are called clans (see [7] for the origin of this name). The definition is the following:

Definition 1.5 ([7]) *Let* (B, C) *be a contact algebra. A subset* $\Gamma \subseteq B$ *is called a clan if the following conditions are satisfied:*
(i) $1 \in \Gamma$ *and* $0 \notin \Gamma$*,*
(ii) If $a \in \Gamma$ *and* $a \leq b$ *then* $b \in \Gamma$*,*
(iii) If $a + b \in \Gamma$ *then* $a \in \Gamma$ *or* $b \in \Gamma$*,*
(iv) If $a, b \in \Gamma$ *then* aCb*.*

The set of all clans is denoted by $CLANS(B)$. For $a \in B$ we denote by $h(a) = \{\Gamma \in CLANS(B) : a \in \Gamma\}$. It is shown in [7] that the set $CLANS(B)$

with the set $\{h(a) : a \in B\}$ considered as a closed base of a topology in CLANS(B), defines a semiregular and compact T_0 topology and h is the embedding of (B, C) into $RC(CLANS(B))$ considered in Theorem 1.3. The following lemma, used in the proof of Theorem 1.3, characterizes contact relation on terms of clans.

Lemma 1.6 *aCb iff $(\exists\Gamma \in CLANS(B))(a, b \in \Gamma)$ iff $h(a) \cap h(b) \neq \varnothing$ [7].*

Precontact algebras. A slight generalization of contact algebra is the notion of *precontact algebra*, studied in [8] under the name *proximity algebra*. The definition sounds as follows.

Definition 1.7 (Precontact algebras) *Let $\underline{B}$ be a Boolean algebra and C is a binary relation on B. C is called a* precontact *relation on B if it satisfies the following conditions:*

(P1) $aCb \to a \neq 0$ and $b \neq 0$,

(P2a) $aC(b + c) \leftrightarrow aCb$ or aCc,

(P2b) $(a + b)Cc \leftrightarrow aCc$ or bCc.

The pair (B, C) is called a precontact algebra.

Obviously every contact relation is a precontact relation.

The following property is an easy consequence from the axioms:

(Mono C) $aCb \wedge a \leq a' \wedge b \leq b' \to a'Cb'$.

2 Dynamic models of space with explicit moments of time and time ordering

Choosing the right definition. In this section we will present a dynamic model of time with explicitly given time moments and precedence relation between them. The construction is based on the following intuition. Suppose we have a domain with changing regions in time and a camera making snapshots for each moment of time. Then each snapshot describes the picture of the state of affairs in the corresponding moment of time m. We assume that the regions at each moment m form a contact algebra - $\underline{B}_m$, which describes their spatial interrelations. In this way to each m from a given set T of time moments we associate a contact algebra $\underline{B}_m$. Each changing region has a trajectory, which can be considered as a vector with coordinates indexed by the elements of T, and each coordinate a_m being from the contact algebra $\underline{B}_m$. We identify changing regions by their trajectories and assume that they form a Boolean algebra with Boolean operations defined coordinate-wise. In this way they form a Boolean subalgebra of the cartesian product $\prod_{m \in T} \underline{B}_m$ of the family of contact algebras $\{\underline{B}_m : m \in B_m\}$. We may assume also that T is supplied with some natural ordering relation $\prec$.

The above informal reasoning suggests the following formal definition.

Definition 2.1 (Dynamic models of space with explicit moments of time and time ordering) *Let T be a non-empty set, which elements are called time moments and $\prec$ be a binary relation on T, called* time ordering*, or* before-after *relation ($m \prec n$ is read: m is before n, or n is after m).*

Then the system $\underline{T} = (T, \prec)$, $T \neq \varnothing$ *is called a* time structure. *Let for each* $m \in T$, $(\underline{B}_m, C_m) = (\underline{B}_m, 0_m, 1_m, \leq_m, ., +, *, C_m)$ *be a contact algebra. Let* $B(\underline{T}) = \prod_{m \in T} B_m$ *denote the Cartesian product of the family* $\{\underline{B}_m : m \in T\}$ *considering its members as Boolean algebras. Since contact algebras are non-degenerated Boolean algebras, then their product* $B(\underline{T})$ *is also a non-degenerated Boolean algebra (note that this fact does not depend on the Axiom of Choice). The algebras* $(\underline{B}_m, C_m)$, $m \in T$, *are called* coordinate contact algebras *of* $B(\underline{T})$. *The elements of* $B(\underline{T})$ *are vectors with coordinates in the coordinate contact algebras, so they can be considered as possible trajectories of changing regions. So we name the elements of* $B(\underline{T})$ *dynamic regions. The product* $B(\underline{T})$ *contains all possible trajectories of changing regions, which is an extreme case, so it is natural to consider subalgebras of the product. Any such subalgebra is called a* dynamic model of space with explicit moments of time and time ordering, *shortly* dynamic model of space.

Let **B** *be a dynamic model of space.* **B** *is called* full *if it coincides with the product* $B(\underline{T})$*;* **B** *is called* rich *if it contains all vectors in which there are no coordinates different from zero and one. Obviously full models are rich.*

In dynamic model of space time moments and the precedence relation are given explicitly by the time structure $(T, \prec)$. By the topological representation theory of contact algebras each coordinate contact algebra $(\underline{B}_m, C_m)$ determines its topological space X_m, called *coordinate space*. By the joint embedding theorem 1.4 all coordinate algebras can be embedded into the contact algebra of regular closed sets $RC(X)$ of a single space X, which may be considered as the set of points of the dynamic model of space.

We will use the following notations for dynamic regions. If $a \in B(\underline{T})$ and $m \in T$, then a_m denotes the m-th coordinate of a and a_m is *the region a at the moment m*. For instance the expression $a_m \neq 0_m$ means that *a exists at m*, and the expression $a_m C_m b_m$ means that *a and b are in a contact at the moment m*.

Dynamic model of space is a quite rich spatio-temporal structure in which one can give explicit definitions of various spatio-temporal relations between dynamic regions. In this paper we shall study the following three relations between dynamic regions:

- $aC^s b$ iff $(\exists m \in T)(a_m C_m b_m)$, called *space contact*,
- $aC^t b$ iff $(\exists m \in T)(a_m \neq 0_m$ and $b_m \neq 0_m)$, called *time contact*,
- $a\mathcal{B}b$ iff $(\exists m, n \in T)(m \prec n$ and $a_m \neq 0_m$ and $b_n \neq 0_n)$, called *precedence*.

While space contact means having a common space point, the time contact indeed means having a common time point. It also is a kind of *simultaneity relation* or *contemporaneity* relation used in Whitehead's works.

Definition 2.2 *Dynamic model of space supplied with the three relations* C^s, C^t, *and* $\mathcal{B}$ *is called a* standard dynamic contact algebra. *It is called* rich *if the dynamic model is rich.*

Our aim is to give abstract point-free characterization of standard dynamic contact algebras. First we will study some formal properties of the three relations, which in the abstract setting will be taken as axioms.

Lemma 2.3 *(i)* C^s *is a contact relation,*
(ii) C^t *is a contact relation satisfying the additional condition*
$(C^s \rightarrow C^t)$ $aC^sb \rightarrow aC^tb$.
(iii) $\mathcal{B}$ *is a* precontact relation*.*

Proof. Direct verification. □

Correspondence results for time structures. Below we list some correspondences between conditions on time ordering $\prec$ and conditions on dynamic regions in standard dynamic contact algebras formulated in terms of C^t and $\mathcal{B}$.
(RS) *Right seriality* $(\forall m)(\exists n)(m \prec n) \iff$ **(rs)** $aC^tb \rightarrow a\mathcal{B}p \vee b\mathcal{B}p^*$,
(LS)*Left seriality* $(\forall m)(\exists n)(n \prec m) \iff$ **(ls)** $aC^tb \rightarrow p\mathcal{B}a \vee p^*\mathcal{B}b$,
(Up Dir) *Updirectness* $(\forall i,j)(\exists k)(i \prec k$ and $j \prec k) \iff$
(up dir) $aC^tb \wedge cC^td \wedge a' + b' + c' + d' = 1 \rightarrow a\mathcal{B}a' \vee b\mathcal{B}b' \vee c\mathcal{B}c' \vee d\mathcal{B}d'$,
(Down Dir) *Downdirectness* $(\forall i,j)(\exists k)(k \prec i$ and $k \prec j) \iff$
(down dir) $aC^tb \wedge cC^td \wedge a' + b' + c' + d' = 1 \rightarrow a'\mathcal{B}a \vee b'\mathcal{B}b \vee c'\mathcal{B}c \vee d'\mathcal{B}d$,
(Dens) *Density* $i \prec j \rightarrow (\exists k)(i \prec k \wedge k \prec j) \iff$ **(dens)** $a\mathcal{B}b \rightarrow a\mathcal{B}p \vee p^*\mathcal{B}b$,
(Ref) *Reflexivity* $(\forall m)(m \prec m) \iff$ **(ref)** $aC^tb \rightarrow a\mathcal{B}b$,
(Lin) *Linearity* $(\forall m,n)(m \prec n \vee n \prec m) \iff$ **(lin)** $aC^tb \wedge cC^td \rightarrow a\mathcal{B}c \vee d\mathcal{B}b$,
(Tr) *Transitivity* $i \prec j$ *and* $j \prec k \rightarrow i \prec k \iff$ **(tr)** $a\overline{\mathcal{B}}b \rightarrow (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$.

Note 1 The correspondence $\ldots \iff \ldots$ means that the condition from the left side is universally true in the time structure $(T,\prec)$ iff the condition from the right side is universally true in **B**. Note that the above listed conditions for time ordering are not independent, for instance (Ref) implies (RS), (LS) and (Dens); (Tr) implies (Up Dir) and (Down Dir). Taking some meaningful subsets of these conditions we obtain various different notions of time order. For instance the subsets $\{(Ref),(Tr),(Lin)\}$ and $\{(RS),(LS),(Tr),(Lin),(Dens)\}$ are typical for classical (reflexive or non-reflexive) time, while the subsets $\{(Ref),(Tr),(UpDir),(DownDir)\}$ or $\{(RS),(LS),(Tr),(UpDir),(DownDir)\}$ are used to characterize some types of relativistic time (for the later see, for instance, [12,18,17,16]). Note that irreflexivity of $\prec$ and linearity for irreflexive $\prec$ in the form $m = n \vee m \prec n \vee n \prec m$ are also good properties of time ordering, but we did not find suitable correspondence conditions for them.

Lemma 2.4 (Correspondence Lemma) *Let* **B** *be a standard dynamic contact algebra with time structure* $(T,\prec)$*, and consider the above listed correspondences except transitivity. They are true under the following conditions: (1) for the implication from left to the right* **B** *is arbitrary, (2) for the implication from right to the left* **B** *is supposed to be rich. (3) for the case of transitivity in both directions* **B** *is supposed to be rich.*

Proof. See Appendix A. □

3 Dynamic contact algebras

Definition 3.1 *By a dynamic contact algebra (DCA for short) we mean any system* $\underline{B} = (B, C^s, C^t, \mathcal{B}) = (B, 0, 1, ., +, ^*, C^s, C^t, \mathcal{B})$ *where* $(B, 0, 1, ., +, ^*)$ *is a non-degenerate Boolean algebra, satisfying the following conditions:*

(i) C^s *is a contact relation on* B *called* space contact,

(ii) C^t *is a contact relation on* B, *called* time contact *satisfying the following additional axiom*

$(C^s \to C^t)$ $aC^sb \to aC^tb$.

(iii) $\mathcal{B}$ *is a precontact relation.*

The elements of B *are called dynamic regions. We will consider DCA-s satisfying some of the eight conditions (rs), (ls),(up dir), (down dir), (ref), (lin), (dens) and (tr) listed in Section 2. Since these axioms determine the properties of the time ordering between moments of time (which will be defined in the next section), they are called shortly* "time axioms". *Since DCA-s are algebraic systems, we adopt for them the standard definitions of subalgebra, homomorphism, isomorphism, isomorphic embedding, etc.*

Note 2 In [23] the name "dynamic contact algebra" (DCA) was used for another, but similar notion. We consider DCA as an integral name for a wider class of algebras formalizing some aspects of an integrated point-free theory of space and time. In this paper, however, it will be used as given by Definition 3.1.

Typical examples of DCA-s are standard dynamic contact algebras introduced in Section 2. In the next section we will show that each DCA is isomorphic to an algebra of such a kind.

Ultrafilter characterizations of the relations C^s, C^t **and** $\mathcal{B}$. Let $\underline{B}$ be a DCA. We denote by $UF(\underline{B})$ the set of ultrafilters in $\underline{B}$. We define three relations $R^s, R^t, \prec$ between ultrafilters as follows: UR^sV iff $U \times V \subseteq C^s$, UR^tV iff $U \times V \subseteq C^t$, $U \prec V$ iff $U \times V \subseteq \mathcal{B}$. Note that these three definitions are meaningful not only for ultrafilters but also for arbitrary subsets of B.

Lemma 3.2 *Let* C *denote any of the three relations* $C^s, C^t, \mathcal{B}$ *and let* R *be the above defined corresponding relation between filters. Then:*

(i) If F, G *are filters and* FRG, *then there are ultrafilters* U, V *such that* $F \subseteq U$, $G \subseteq V$ *and* URV.

(ii) aCb *iff there exist ultrafilters* U, V *such that* URV, $a \in U$ *and* $b \in V$.

(iii) R^s *and* R^t *are reflexive and symmetric,*

(iv) $R^s \subseteq R^t$

Proof. Conditions (i), (ii) and (iii) are from [8] and (iv) follows directly from axiom $(C^s \to C^t)$. □

Now we list some correspondences between the eight conditions (rs)– (tr) listed in Section 2 and corresponding conditions between ultrafilter relations R^t and $\prec$. They are similar to the corresponding relations (RS)–(Tr). We give the new relations the same short notations but in square brackets - [RS] – [Tr].

[RS] $U_1R^tU_2 \to (\exists V)(U_1 \prec V \wedge U_2 \prec V) \Longleftrightarrow$ **(rs)** $aC^tb \to a\mathcal{B}p \vee b\mathcal{B}p^*$,
[LS] $U_1R^tU_2 \to (\exists V)(V \prec U_1 \wedge V \prec U_2) \Longleftrightarrow$ **(ls)** $aC^tb \to p\mathcal{B}a \vee p^*\mathcal{B}b$,
[Up Dir] $U_1R^tU_2 \wedge U_3R^tU_4 \to (\exists V)(U_1 \prec V \wedge U_2 \prec V \wedge U_3 \prec V \wedge U_4 \prec V)$
$\Longleftrightarrow$ **(up dir)** $aC^tb \wedge cC^td \wedge a' + b' + c' + d' = 1 \to a\mathcal{B}a' \vee b\mathcal{B}b' \vee c\mathcal{B}c' \vee d\mathcal{B}d'$,
[Down Dir] $U_1R^tU_2 \wedge U_3R^tU_4 \to (\exists V)(V \prec U_1 \wedge V \prec U_2 \wedge V \prec U_3 \wedge V \prec U_4)$
$\Longleftrightarrow$ **(down dir)** $aC^tb \wedge cC^td \wedge a' + b' + c' + d' = 1 \to a'\mathcal{B}a \vee b'\mathcal{B}b \vee c'\mathcal{B}c \vee d'\mathcal{B}d$,
[Dens] $U \prec V \to (\exists W)(U \prec W \wedge W \prec V) \Longleftrightarrow$ **(dens)** $a\mathcal{B}b \to a\mathcal{B}p \vee p^*\mathcal{B}b$,
[Ref] $UR^tV \to U \prec V \Longleftrightarrow$ **(ref)** $aC^tb \to a\mathcal{B}b$,
[Lin] $U_1R^tU_2 \wedge V_1R^tV_2 \to (U_1 \prec V_1 \vee V_2 \prec U_2) \Longleftrightarrow$
(lin) $aC^tb \wedge cC^td \to a\mathcal{B}c \vee d\mathcal{B}b$,
[Tr] $U \prec V \wedge V \prec W \to U \prec W \Longleftrightarrow$ **(tr)** $a\overline{\mathcal{B}}b \to (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$.

Lemma 3.3 (Ultrafilter correspondences) *Let $\underline{B} = (B, C^s, C^t, \mathcal{B})$ be a DCA. Then all eight correspondences listed above are true in the following sense: for a given correspondence ... $\Longleftrightarrow$ ..., the left side is universally true in the set $UF(\underline{B})$) iff the right side is universally true in $\underline{B}$.*

Proof. See Appendix B. □

4 Representation theory of dynamic contact algebras

In this section we will show that each DCA, with possible extension with some additional time axioms, can be represented as a dynamic model of space with explicit moments of time and time ordering, satisfying some reasonable properties. The representation theorem will be based on a canonical construction of the dynamic model of space associated to the given DCA $\underline{B} = (B, C^s, C^t, \mathcal{B})$. Our strategy is the following. First we define the set $T = T(\underline{B})$ of moments of time and before-after relation $\prec$ in T. The time moments are defined as some sets of ultrafilters in T with a canonical definition of $\prec$ with properties determined by the corresponding time axioms. The next step is to associate to each time moment the corresponding coordinate contact algebra and to define the canonical dynamic model of space and the canonical embedding isomorphism. The final step is to show that it indeed embeds the algebra into the obtained dynamic model of space.

Defining the time structure.

Definition 4.1 *Let $\underline{B} = (B, C^s, C^t, \mathcal{B})$ be a DCA. A set α of ultrafilters in $\underline{B}$ is called a* time moment *if it satisfies the following conditions:*

(tm1) *α is non-empty,*

(tm2) *If $U, V \in \alpha$, then UR^tV,*

(tm3) *If $\underline{B}$ satisfies some of the axioms (rs),(ls), (up dir), (down dir) and (dens) then α has at most two different ultrafilters.*

*The set of time moments of $\underline{B}$ is denoted by $T(\underline{B})$ or simply by T. Obviously T is nonempty, because, for instance, each singleton set $\{U\}$, where U is an ultrafilter, is a time point (condition **(tm1)** is obviously satisfied and **(tm2)** is satisfied because of the reflexivity of R^t).*

The definition of time ordering $\prec$ *is this: for $\alpha, \beta \in T$*

(to) *$\alpha \prec \beta$ iff $(\forall U, V \in UF(\underline{B}))(U \in \alpha \wedge V \in \beta \to U \prec V)$.*

The pair $\underline{T}(\underline{B}) = (T(\underline{B}), \prec)$ (denoted sometimes shortly $(T, \prec)$) is called the time structure of $\underline{B}$.

Note that condition **(tm3)** is taken only by technical reasons for the proofs of the properties of time ordering, and depends on the fact that the correspondents of the axioms mentioned in the condition, namely [RS], [LS], [Up Dir], [Down Dir] and [Dens], are not universal sentences, while the correspondents of the other time axioms are universal sentences.

Lemma 4.2 (Properties of time ordering) *Let $\underline{B} = (B, C^s, C^t, \mathcal{B})$ be a dynamic contact algebra and $(T, \prec)$ be its time structure. Consider the correspondences $\ldots \Longleftrightarrow \ldots$ from Section 2. Then each correspondence is true in the following sense: the condition from the left side of $\ldots \Longleftrightarrow \ldots$ is universally true in the time structure $(T, \prec)$) iff the right side is universally true in the algebra $\underline{B}$.*

Proof. See Appendix C. □

Defining coordinate contact algebras, the canonical dynamic model of space and the isomorphic embedding. To define coordinate contact algebras we will use the following construction of *factor contact algebra* from sets of ultrafilters in a given contact algebra.

Let α be a non-empty set of ultrafilters in a contact algebra $\underline{B} = (B, C)$. Define the following equivalence relation on B depending on α:

$a \equiv_\alpha b$ iff $(\forall U \in \alpha)(a \in U \leftrightarrow b \in U)$.

It is easy to see that $\equiv_\alpha$ is a congruence on $\underline{B}$ and let $\underline{B}/\alpha$ denote the factor algebra $\underline{B}/\equiv_\alpha$. Denote the Boolean ordering on $\underline{B}/\alpha$ by $\leq_\alpha$. Define the relation $|a|_\alpha C_\alpha |b|_\alpha$ iff there exist ultrafilters $U, V \in \alpha$ such that URV and $a \in U$ and $b \in V$, ($URV \leftrightarrow_{def} U \times V \subseteq C$, see Lemma 3.2).

Lemma 4.3 *(i) B_α is a non-degenerated Boolean algebra.*

(ii) $|a|_\alpha \neq |0|_\alpha$ iff there exists an ultrafilter $U \in \alpha$ such that $a \in U$.

(iii) The relation C_α is well defined and the pair (B_α, C_α) is a contact algebra.

Proof. Direct verification. □

Definition 4.4 (Coordinate contact algebras, the canonical dynamic model of space and the embedding) *Let $\underline{B}$ be a DCA and α be a moment of time in $\underline{B}$. Define the factor Boolean algebra $\underline{B}_\alpha$ by the construction described above and the contact relation*

$|a|_\alpha C_\alpha |b|_\alpha$ iff $(\exists U, V \in \alpha)(UR^sV,\ a \in U$ and $b \in V)$.

Then by Lemma 4.3 $(\underline{B}_\alpha, C_\alpha)$ is a contact algebra, called the coordinate contact algebra associated to α. The canonical dynamic model of space, denoted here $\mathbf{B}^{can}$, and the relations C^s, C^t and $\mathcal{B}$ in it are defined by the just defined coordinate contact algebras as in Section 2. The embedding h is defined coordinatewise as follows: for each $a \in B$,

$h(a)_\alpha = |a|_\alpha$.

Note 3 The definition of the coordinate contact algebra $\underline{B}_\alpha$ as a factor algebra with respect to the set of ultrafilters of the time moment α is based on the following intuition. If we look at dynamic regions as trajectories of changing regions, then for different a and b we may have that $a_\alpha = b_\alpha$, which is an equivalence relation determined by α. The formal definition of this equivalence is just the relation $\equiv_\alpha$, which determines the coordinate contact algebra $\underline{B}_\alpha$. So the elements of $\underline{B}_\alpha$ are the equivalence classes $|a|_\alpha$.

Lemma 4.5 (Embedding Lemma) *Let $\underline{B}$ be a DCA. Then:*

(i) aC^sb iff there exists $\alpha \in T$ such that $|a|_\alpha C_\alpha |b|_\alpha$ iff $h(a)C^sh(b)$,

(ii) aC^tb iff there exists $\alpha \in T$ such that $|a|_\alpha \neq |0|_\alpha$ and $|b|_\alpha \neq |0|_\alpha$ iff $h(a)C^th(b)$,

(iii) $a\mathcal{B}b$ iff there exist $\alpha, \beta \in T$ such that $\alpha \prec \beta$ and $|a|_\alpha \neq |0|_\alpha$ and $|b|_\beta \neq |0|_\beta$ iff $h(a)\mathcal{B}h(b)$,

(iv) $a \leq b$ iff for all $\alpha \in T$ $|a|_\alpha \leq_\alpha |b|_\alpha$ iff $h(a) \leq h(b)$,

(v) h preserves Boolean operations.

Proof. See Appendix D. □

Lemma 4.6 *Let $\underline{B}$ be a DCA and $\mathbf{B}^{can}$ be its canonical dynamic model of space. Then for each time axiom Ax from the list $\langle (rs), (ls), \ldots, (tr)\rangle$ the following equivalence is true: Ax holds in $\underline{B}$ iff Ax holds in $\mathbf{B}^{can}$.*

Proof. The proof for all cases for Ax is the same, so we will illustrate it for $Ax = (rs)$. Namely we have the following chain of equivalencies:
The condition (rs) holds in $\underline{B} \Longleftrightarrow$ (by Lemma 4.2) the condition (RS) holds in the time structure $(T, \prec)$ of $\underline{B} \Longleftrightarrow$ (by Lemma 2.4) the condition (rs) holds in $\mathbf{B}^{can}$. □

Theorem 4.7 (Representation Theorem for DCA-s) *Let $\underline{B}$ be a DCA. Then there exists a dynamic model of space $\mathbf{B}$ and an isomorphic embedding h of $\underline{B}$ into $\mathbf{B}$. Moreover, $\underline{B}$ satisfies some of the time axioms iff the same axioms are satisfied in $\mathbf{B}$.*

Proof. The proof is a direct corollary of Lemma 4.5 and Lemma 4.6. □

5 Quantifier-free logics based on dynamic contact algebras

In this section we will present a complete axiomatization of some quantifier-free logics based on the language of dynamic contact algebras. The motivation to consider quantifier-free logics is to obtain decidable fragments. The language contains a denumerable set of variables, called Boolean variables, two symbols for the Boolean constants 0 and 1, symbols $+, ., ^*$ for the Boolean operations, equality $=$ and three two place predicate symbols $C^s, C^t, \mathcal{B}$. Terms of this language (called Boolean terms) are build in the standard way from Boolean variables and constants. Atomic formulas are of the form $a = b$, aC^sb, aC^tb and $a\mathcal{B}b$, where a, b are Boolean terms. Formulas are build from the atomic formulas by means of the propositional operations $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ in a standard way.

The intended semantics of the introduced language is in various classes of DCA-s. Let $\underline{B}$ be a DCA, a valuation v in $\underline{B}$ is a mapping from the set of Boolean variables and constants to B extended in a standard way to the set of all terms. The pair (B, v) is called a model. The truth of a formula α in the model (B, v), denoted by $(B, v) \models \alpha$ is defined inductively as follows:

$(B, v) \models a = b$ iff $v(a) = v(b)$,

$(B, v) \models aC^s b$ iff $v(a)C^s v(b)$, and similarly for $aC^t b$ and $a\mathcal{B}b$.

$(B, v) \models \neg\alpha$ iff $(B, v) \not\models \alpha$,

$(B, v) \models \alpha \wedge \beta$ iff $(B, v) \models \alpha$ and $(B, v) \models \beta$,

and similarly for the other propositional connectives.

We say that α is true in the algebra $\underline{B}$ if it is true in all models over $\underline{B}$; α is true in a class Σ of DCA-s if it is true in all algebras from Σ. A formula α has a model in a given class of DCA-s if there is a model in this class in which α is true. A set A of formulas has a model in a given class of DCA-s if there is a model in that class in which all members of A are true.

We denote by $\mathbb{L}_{all}$ the logic corresponding to the class of all DCA-s. Since all axioms of DCA are universal formulas, axiomatization of $\mathbb{L}_{all}$ can be done quantifier-free on the base of propositional logic as follows:

Axiom system for $\mathbb{L}_{all}$.

(I) Axiom schemes for classical propositional logic

Here one can use any Hilbert-style axiomatization of classical propositional logic with axiom schemes with metavariables ranging in the set of formulas.

(II) Axiom schemes for Boolean algebra with the axioms of equality.

Since Boolean algebras can be axiomatized by universal formulas in a first-order logic with equality, one can take any such set of axiom schemes plus the axioms of equality, all written in a quantifier-free form.

(III) Specific axioms for DCA

Since the axioms of DCA are all universal sentences, we rewrite them as formulas of our language

Rules of inference: The only rule is Modus Ponens (MP) $\frac{\alpha, \alpha \Rightarrow \beta}{\beta}$

Since all time axioms except the axiom (tr) are also universal formulas, extensions of $\mathbb{L}_{all}$ with such axioms can be done just by adding them as axiom schemes to the above axiom system. Since the axiom of transitivity (tr) is not a universal formula, we can not add it to the axiom system of $\mathbb{L}_{all}$, but instead we can add the following additional rule of inference, which in a sense imitates the corresponding axiom:

The rule of transitivity TR: $\dfrac{\alpha \Rightarrow (a\mathcal{B}p \vee p^*\mathcal{B}b)}{\alpha \Rightarrow a\mathcal{B}b}$, where p is a Boolean variable that does not occur in a, b, and α.

If Ax is a set of time axioms then $\mathbb{L}_{Ax}$ will denote the extensions of $\mathbb{L}_{all}$ with the axiom schemes from Ax, where for the case of transitivity axiom (tr) we consider the rule of transitivity **TR**.

Let us note that the rule of transitivity **TR** was studied in [1] under the name of **rule of normality (NOR)**.

The following theorem is the main result in this section:

Theorem 5.1 (Completeness theorem for $\mathbb{L}_{Ax}$) *The logic $\mathbb{L}_{Ax}$ is strongly sound and complete in the class of all dynamic contact algebras satisfying the axioms Ax.*

Proof. See appendix E. □

Let us note that due to the representation Theorem 4.7 for dynamic contact algebras we obtain as a corollary from Theorem 5.1 also completeness of $\mathbb{L}_{Ax}$ with respect to the corresponding class of standard dynamic contact algebras.

Theorem 5.2 (Decidability of some $\mathbb{L}_{Ax}$) *Let Ax be a subset of time axioms not containing the axiom (tr). Then the logic $\mathbb{L}_{Ax}$ is decidable.*

Proof. We shall show that the logic has finite model property. Let α be a formula which is not a theorem. Then by the completeness theorem there is a model (B, v) based on some dynamic contact algebra $\underline{B}$ from the class of algebras in which $\mathbb{L}_{Ax}$ is complete, such that $(B, v) \not\models \alpha$. Let $\underline{B}_{fin}$ be the finite Boolean subalgebra of $\underline{B}$ generated by the set $\{v(b_1), \ldots, v(b_n)\}$, where $b_1, \ldots, b_n$ are all variables of α. Since all axioms from Ax are universal formulas, then they are satisfied in $\underline{B}_{fin}$ and hence it is in the class of algebras in which the logic is complete. So α is falsified in a finite algebra with the size $\leq 2^{2^n}$ where n is the number of the variables of α. □

Let us note that we do not know if decidability is preserved by adding to the logic the rule of transitivity. We formulate also as an open problem the complexity of the decision problems related to discussed logics.

6 Concluding remarks

Related works. Some other works quite similar to our approach are [9,10]. They are point-free with respect to space points but not with respect to time points in the sense that the set of time points is explicitly given in the axiomatization. A point-free version of *dynamic mereology* based on some natural stable and unstable mereological relations is [14]. The survey paper [13] is devoted to various combinations of spatial and temporal logics, concerning mainly expressivity and complexity of formalisms, but not point-free representations (see also [2,11]). Modal logics for Minkowski space-time, based on different ideas, are considered in [12,18,17,16].

Discussion and open problems. We want to note here that this "second attempt" to present an integrated theory of space and time in an Whiteheadian manner shows that the structure of time can also be characterized in a point-free way considering as primitives neither time points, nor their intended internal ordering structure. The three primitive spatio-temporal relations between dynamic regions – space contact C^s, time contact C^t and precedence $\mathcal{B}$ studied in the present paper form in a sense the minimal set of primitives which guarantee the definitions of space points and the coordinate contact algebras and the set time points and the corresponding time structure $(T, \prec)$. There are, however, many other spatio-temporal relations which are not considered in this paper. For instance we do not consider stable contact $C^{\forall}$, considered in [23]. The reason is that canonical constructions used in the representation theory in [23] are not compatible with the canonical constructions used in the present paper and it is an open problem to characterize $C^{\forall}$ in the context of the primitives C^t and $\mathcal{B}$. So we formulate as an open problem the extension of the language of dynamic contact algebras with other sensible spatio-temporal relations between dynamic regions depending on time order. For instance we may define in the standard model the following one-place predicate – *"a has only one period of life"* with the following formal definition: there exist two moments of time $m \neq n$ and $m \prec n$ such that for all k in the set $[m, n] =_{def} \{k \in T : k = m \vee k = n \vee m \prec k \prec n\}$, a exists at k (formally $a_k \neq 0_k$) and for all $k \notin [m, n]$, a does not exist". For instance living organisms have only one period of life in the above sense. Another open problem is to ensure coordinate contact algebras to determine some good classes of topological spaces: connected spaces, regular spaces, Hausdorff spaces, Euclidean spaces.

Let us mention that the time contact C^t corresponds to the natural notion of *simultaneity* or *contemporaneity*. It can be generalized for any finite set A of regions: $CON(A)$ (the members of A are contemporaries) iff there exists $m \in T$ such that for all $a \in A$, $a_m \neq 0_m$ (a_m exists at the moment m). Such a polyadic version is considered also by Whitehead and we plan to axiomatize this generalization in an extended version of this paper. Let us note, however, that our notion of contemporaneity, which is based on its standard use in the ordinary language, differs from the meaning used by Whitehead, who considered it as a kind of *"causal independency"* (introduced rather informally). Causal independency in Whitehead is related to some other relations named *"causal future"* and *"causal past"*, which are influenced by relativity theory (see [24], part IV). The exact definitions of similar causal relations, considered as relations between points in Minkowski space, are studied in some modal logics of space-time [12,18,17,16]. It will be nice to have good formal analogs of such relations considered as relations between dynamic regions.

We plan to apply a translation of the quantifier-free logics studied in this paper in suitable modal logics with universal modality in order to see if the rule of transitivity **TR** can be eliminated and to use this fact for the study of decidability and complexity problems related to these logics.

Thanks. The paper is sponsored by the contract with the Bulgarian National Science Fund, project name: "Theories of Space and Time: algebraic, topological and logical approaches". Thanks are due to the anonymous referees for their useful suggestions and also to Philippe Balbiani and Tinko Tinchev for their fruitful comments and discussion on an earlier version of this paper. I am indebted also to Veselin Petrov, a Bulgarian philosopher and mathematician, who introduced me to Whitehead's philosophy.

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Appendix A: proof of Lemma 2.4 (Correspondence Lemma)

Proof. We consider first the case of transitivity. Suppose that **B** is rich.

(**Tr**) $\Longrightarrow$ (**tr**) Suppose that $\prec$ is transitive and let $a\overline{\mathcal{B}}b$. We have to find c such that ($a\overline{\mathcal{B}}c$ and $c^*\overline{\mathcal{B}}b$). Define c coordinatewise as follows:

$$c_k = \begin{cases} 1_k, & \text{if } (\exists j)(k \prec j \wedge b_j \neq 0_j) \\ 0_k, & \text{if } (\exists i)(i \prec k \wedge a_i \neq 0_i) \end{cases}$$

The definition of c_k is correct, because if the two conditions are satisfied simultaneously then by transitivity we obtain $i \prec j$ which together with $a_i \neq 0_i$ and $b_j \neq 0_j$ imply $a\mathcal{B}b$ - a contradiction. Obviously by richness $c \in \mathbf{B}$.

To show $a\overline{\mathcal{B}}c$ suppose the contrary. Then there exist $i, k \in T$ such that $i \prec k$, $a_i \neq 0_i$ and $c_k \neq 0_k$. From here and by the definition of c_k we get $c_k = 1_k$. By the definition of c_k, this implies that there exists $j \in T$ such that $k \prec j$ and $b_j \neq 0_j$. From $i \prec k$ and $k \prec j$ we obtain (by transitivity) $i \prec j$, which together with $a_i \neq 0_i$ and $b_j \neq 0_j$ imply $a\mathcal{B}b$ - a contradiction with the assumption $a\overline{\mathcal{B}}b$.

To show $c^*\overline{\mathcal{B}}b$ suppose again the contrary. Then there exists $k, j \in T$ such that $k \prec j$, $c^*_k \neq 0_k$ and $b_j \neq 0_j$. From here we get $c^*_k = 1_k$ and $c_k = 0_k$. This implies that there exists $i \in T$ such that $i \prec k$ and $a_i \neq 0_i$. Conditions $i \prec k$ and $k \prec j$ imply $i \prec j$ which together with $a_i \neq 0_i$ and $b_j \neq 0_j$ imply $a\mathcal{B}b$ - again a contradiction with the assumption $a\overline{\mathcal{B}}b$.

(**Tr**) $\Longleftarrow$ (**tr**) Suppose that transitivity does not hold for $\prec$, then for some i', j', k' we have $i' \prec j'$, $j' \prec k'$ but $i' \not\prec k'$. Define a and b coordinatewise as follows:

$$a_i = \begin{cases} 1_i, & \text{if } i = i' \\ 0_i, & \text{if } i \neq i' \end{cases}, \; b_k = \begin{cases} 1_k, & \text{if } k = k' \\ 0_k, & \text{if } k \neq k' \end{cases}.$$

By richness a and b belong to **B**. We will show that $a\overline{\mathcal{B}}b \rightarrow (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$ does not hold, i.e. $a\overline{\mathcal{B}}b$ and $\neg(\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$.

First we show $a\overline{\mathcal{B}}b$. Suppose the contrary. Then for some $i, k \in T$ we have $i \prec k$, $a_i \neq 0_i$ and $b_k \neq 0_k$. This implies $i = i'$, $k = k'$ and hence $i' \prec k'$ - a contradiction with the assumption $i' \not\prec k'$.

Now we will show $\neg(\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$. Suppose the contrary, i.e. that there exists c such that $a\overline{\mathcal{B}}c$ and $c^*\overline{\mathcal{B}}b$.

Case 1. $c_{j'} \neq 0_{j'}$. We have $i' \prec j'$, $a_{i'} = 1_{i'} \neq 0_{i'}$. This implies $a\mathcal{B}c$ - a contradiction with the assumption $a\overline{\mathcal{B}}c$.

Case 2. $c_{j'} = 0_{j'}$. Then $c^*_{j'} = 1_{j'} \neq 0_{j'}$. We have $j' \prec k'$ and $b_{k'} = 1_{k'} \neq 0_{k'}$. This implies $c^*\mathcal{B}b$ - a contradiction with the assumption $c^*\overline{\mathcal{B}}b$.

(**RS**) $\Longrightarrow$ (**rs**) Suppose that $\prec$ is right-serial and let aC^tb. Then for some $m \in T$ we have $a_m \neq 0_m$ and $b_m \neq 0_m$. By right-seriality there exists $n \in T$ such that $m \prec n$. Let p be an arbitrary region.

Case $p_n \neq 0_n$**.** This implies $a\mathcal{B}p$.

Case $p_n = 0_n$**.** This implies $p_n^* \neq 0_n$ and hence $b\mathcal{B}p^*$.

So, in both cases we obtain $aC^t b \to a\mathcal{B}p \vee b\mathcal{B}p^*$.

(**RS**) $\Longleftarrow$ (**rs**) In this case we will reason by contraposition. Suppose that $\prec$ is not right-serial so $(\exists m')(\forall n)(m' \not\prec n)$. Under this assumption we will proceed to show that $aC^t b \to a\mathcal{B}p \vee b\mathcal{B}p^*$ does not hold, so for some a, b, p $aC^t b$, $a\overline{\mathcal{B}}p$ and $b\overline{\mathcal{B}}p^*$. Let $p = 1$ and define a, b coordinatewise as follows:

$$a_m = b_m = \begin{cases} 1_m, & \text{if } m = m' \\ 0_m, & \text{if } m \neq m' \end{cases},$$

By richness a, b are in **B**.

First we will show $aC^t b$. Observe that we have $a_{m'} = b_{m'} = 1_{m'} \neq 0_{m'}$ which implies $aC^t b$.

To show $a\overline{\mathcal{B}}p$ suppose the contrary, i.e. that $a\mathcal{B}p$, i.e. $a\mathcal{B}1$ holds. Then for some $m \prec n$ we have $a_m \neq 0_m$ and $1_n \neq 0_n$. By the definition of a we obtain $a_m = 1_m$ and hence $m = m'$, which implies $m' \prec n$. This contradicts the assumption $(\forall n)(m' \not\prec n)$.

Since $p = 1$, then $p^* = 0$ and hence the condition $b\overline{\mathcal{B}}p^*$ trivially holds.

The proofs for the other cases of the lemma are similar. □

Appendix B: proof of Lemma 3.3 (Ultrafilter correspondences)

Proof. For the proof of Lemma 3.3 we will need the following facts about filters in Boolean algebra and one lemma which we formulated without proofs:
Facts. *If F, G are filters then $F \oplus G =_{def} \{c : (\exists a \in F)(\exists b \in G)(a.b \leq c)\}$ is the smallest filter containing F and G. $F \oplus G$ is not a proper filter (i.e. $0 \in F \oplus G$) iff there exists p such that $p^* \in F$ and $p \in G$. The operation $\oplus$ is associative and commutative. Every proper filter can be extended into an ultrafilter.*

Lemma 6.1 *Let U, V be filters and let $\mathbf{F}_I(U) =_{def} \{b : (\exists a \in U)a\overline{C}b^*\}$ and $\mathbf{F}_{II}(V) =_{def} \{a : (\exists b \in V)a^*\overline{C}b\}$. Then:*

(i) $\mathbf{F}_I(U)$ and $\mathbf{F}_{II}(V)$ are filters,

(ii) If U is a filter and V is an ultrafilter then: $U \times V \leq C$ iff $\mathbf{F}_I(U) \subseteq V$,

(iii If U is an ultrafilter and V is a filter then: $U \times V \leq C$ iff $\mathbf{F}_{II}(V) \subseteq U$.

[RS] $\Rightarrow$ **(rs)**. Suppose [**RS**] holds and let $aC^t b$. Then by Lemma 3.2 There are ultrafilters U_1, U_2 such that $U_1 R^t U_2$, $a \in U_1$ and $b \in U_2$. By [RS] there exists an ultrafilter V such that $U_1 \prec V$ and $U_2 \prec V$. Let p be an arbitrary region.
Case 1: $p \in V$. Then by $U_1 \prec V$ and $a \in U_1$ we conclude by Lemma 3.2 that $a\mathcal{B}p$.
Case 2: $p^* \in V$. As in Case 1, this implies $b\mathcal{B}p^*$.

[RS] $\Leftarrow$ **(rs)**. Suppose **(rs)** holds and let $U_1 R^t U_2$. We shall show that there exists an ultrafilter V such that $U_1 \prec V$ and $U_2 \prec V$.

To prove this we shall show first that $\mathbf{F}_I(U_1) \oplus \mathbf{F}_{II}(U_2)$ is a proper filter (see Lemma 6.1 for notations). Suppose that this is not the case. Then there exists p such that $p^* \in \mathbf{F}_I(U_1)$ and $p \in \mathbf{F}_{II}(U_2)$. This implies that there exists $a \in U_1$ such that $a\overline{\mathcal{B}}p$ and that there exists $b \in U_2$ such that $b\overline{\mathcal{B}}p^*$. Since $U_1 R^t U_2$, this implies $aC^t b$, which by $a\overline{\mathcal{B}}p$ and $b\overline{\mathcal{B}}p^*$ shows that **(rs)** does not hold - a contradiction. Thus $\mathbf{F}_I(U_1) \oplus \mathbf{F}_{II}(U_2)$ is a proper filter. Then it can be extended into an ultrafilter V. Consequently $\mathbf{F}_I(U_1) \subseteq V$ (which implies $U_1 \prec V$) and $\mathbf{F}_{II}(U_2) \subseteq V$ (which implies $U_2 \prec V$), which ends the proof of this case.

The other cases of the lemma can be proved in a similar way making use of Lemma 6.1, Lemma 3.2 and and the above mentioned **Facts** many times. Since (tr) is not an universal sentence, as an illustration we will demonstrate the proof of the implication **[Tr]⇒(tr)**. Another proof of this implication can be found in [8].

[Tr]⇒(tr). Suppose that **[Tr]** holds and let $a\overline{\mathcal{B}}b$. Suppose that there is no c such that $a\overline{\mathcal{B}}c$ and $c^*\overline{\mathcal{B}}b$. We will proceed to obtain a contradiction as follows. First we will show that there are ultrafiltres U, V, W, such that $a \in U$, $b \in W$, $U \prec V$ and $V \prec W$. Then by transitivity we get $U \prec W$. But $a \in U$, $b \in W$ and $U \prec W$ implies $a\mathcal{B}b$ - the desired contradiction.

Now to realize the above strategy define $[a) =_{def} \{a' : a \leq a'\}$, $[b) =_{def} \{b' : b \leq b'\}$. $[a)$ is a filter containing a and $[b)$ is a filter containing b. We shall show that the filter $\mathbf{F}_I([a)) \oplus \mathbf{F}_{II}([b))$ is a proper filter. Otherwise there are $p^* \in \mathbf{F}_I([a))$, $p \in \mathbf{F}_{II}([b))$, which implies (see Lemma 6.1) that there exists $a' \in [a)$ (so $a \leq a'$) such that $a'\overline{\mathcal{B}}p$ and that there exists $b' \in [b)$ (so $b \leq b'$) such that $p^*\overline{\mathcal{B}}b'$. By the monotonicity of $\mathcal{B}$ we obtain $a\overline{\mathcal{B}}p$ and $p^*\overline{\mathcal{B}}b$. This contradicts the assumption that there is no c such that $a\overline{\mathcal{B}}c$ and $c^*\overline{\mathcal{B}}b$ – simply take $c = p$. Consequently $\mathbf{F}_I([a)) \oplus \mathbf{F}_{II}([b))$ is a proper filter. Then it can be extended into an ultrafilter V. Hence we get $\mathbf{F}_I([a)) \subseteq V$ and $\mathbf{F}_{II}([b)) \subseteq V$. This, by 6.1 implies $[b) \times V \subseteq \mathcal{B}$ and $[a) \times V \subseteq \mathcal{B}$. Then applying Lemma 3.2 we can extend $[a)$ into an ultrafilter U such that $U \times V \subseteq \mathcal{B}$,(so $U \prec V$), and similarly to extend $[b)$ into an ultrafilter W such that $V \times W \subseteq \mathcal{B}$, (so $V \prec W$). Obviously $a \in U$ and $b \in W$. Thus we have obtained $U \prec V$, $V \prec W$, $a \in U$ and $b \in W$ - the strategy is fulfilled, which ends the proof of this case. □

Appendix C: proof of Lemma 4.2 (Properties of time ordering)

Proof. The proofs of all cases are similar, so we will demonstrate only two examples: $(Tr) \Longleftrightarrow (tr)$ and $(RS) \Longleftrightarrow (rs)$.

$(Tr) \Longrightarrow (tr)$. Suppose the contrary, i.e. $\prec$ is a transitive relation on T and that (tr) does not hold in $\underline{B}$. Then by Lemma 3.3 there are ultrafilters U, V, W such that $U \prec V$, $V \prec W$ but $U \not\prec W$. Let $\alpha = \{U\}$, $\beta = \{V\}$ and $\gamma = \{W\}$. Since R^t is a reflexive relation, then α, β, γ are time points and hence are elements of T. By the definition of $\prec$ we get $\alpha \prec \beta$, $\beta \prec \gamma$, but not $\alpha \prec \gamma$, which contradicts the transitivity of $\prec$ in T.

$(Tr) \Longleftarrow (tr)$. Suppose that (tr) holds in $\underline{B}$. Then by Lemma 3.3 the relation $\prec$ defined in the set of ultrafilters of $\underline{B}$ is a transitive relation. We will show that $\prec$, as a relation between time moments, is a transitive relation on T. Suppose α, β, γ are from T, $\alpha \prec \beta$, $\beta \prec \gamma$ but $\alpha \not\prec \gamma$. Then there are ultrafilters $U \in \alpha$ and $W \in \gamma$ such that $U \not\prec W$. Since β is non-empty, let $V \in \beta$. Then we obtain $U \prec V$, $V \prec W$, which together with $U \not\prec W$ contradicts the transitivity of $\prec$ in the set of ultrafilters.
$(RS) \Longrightarrow (rs)$. Suppose that the condition **(RS)** holds in T. We will show that $\underline{B}$ satisfies **(rs)**. To this end suppose that aC^tb holds in $\underline{B}$ and proceed to show that ether $a\mathcal{B}p$ holds or $b\mathcal{B}p^*$ holds.

From aC^tb it follows by Lemma 3.2 that there are ultrafilters U, V such that UR^tV, $a \in U$ and $b \in V$. Let $\alpha = \{U, V\}$. Since UR^tV, we obtain that $\alpha \in T$. Then by the condition **(RS)** there exists $\beta \in T$ such that $\alpha \prec \beta$. Let $W \in \beta$. Then by the definition of $\prec$ in T we have $U \prec W$ and $V \prec W$. Let p be an arbitrary region in $\underline{B}$.
Case 1: $p \in W$. We have $a \in U$ and $U \prec W$. This by Lemma 3.2 implies $a\mathcal{B}p$.
Case 2: $p^* \in W$. We have $b \in V$ and $V \prec W$. This by Lemma 3.2 implies $b\mathcal{B}p^*$.
$(RS) \Longleftarrow (rs)$. Suppose that (rs) holds in $\underline{B}$, then by Lemma 3.3 **[RS]** holds in the set of ultrafilters in $\underline{B}$. We shall show that **(RS)** holds in T. Let $\alpha \in T$. Now, since we are working with the condition (rs) the condition **(tm3)** is fulfilled and hence α is in the form $\alpha = \{U_1, U_2\}$. Then by **(tm2)** $U_1R^tU_2$ and by **[RS]** there is an ultrafilter V such that $U_1 \prec V$ and $U_2 \prec V$. Define $\beta = \{V\}$. By the reflexivity of R^t and **(tm2)** we have $\beta \in T$. By the definition of $\prec$ in T we obtain $\alpha \prec \beta$, so **(RS)** holds in T. □

Appendix D: proof of Lemma 4.5 (Embedding Lemma)

Proof. (i)($\Rightarrow$). Suppose aC^sb. By Lemma 3.2 there exist ultrafilters U, V such that UC^sV, $a \in U$ and $b \in V$. Define $\alpha = \{U, V\}$. By Lemma 3.2 we have also UC^tV, so $\alpha \in T$ and hence $|a|_\alpha C_\alpha |b|_\alpha$, which is equivalent to $h(a)C^sh(b)$.

($\Leftarrow$). Suppose $h(a)C^sh(b)$. Then for some $\alpha \in T$ we have $|a|_\alpha C_\alpha |b|_\alpha$ and by the definition of C_α there are ultrafilters $U, V \in \alpha$ such that UR^sV, $a \in U$ and $b \in V$. Then by Lemma 3.2 we obtain aC^sb.

(ii)($\Rightarrow$). Suppose aC^tb. By Lemma 3.2 aC^tb implies that there exist ultrafilters U, V such that UC^tV, $a \in U$ and $b \in V$. Define $\alpha = \{U, V\}$. Since UC^tV we get $\alpha \in T$. By Lemma 4.3 $a \in U \in \alpha$ and $b \in V \in \alpha$ is equivalent to $|a|_\alpha \neq |0|_\alpha$ and $|b|_\alpha \neq |0|_\alpha$, which is equivalent to $h(a)C^th(b)$. Thus aC^tb implies $h(a)C^th(b)$.

($\Leftarrow$). Suppose $h(a)C^th(b)$. This implies that for some $\alpha \in T$, $|a|_\alpha \neq |0|_\alpha$ and $|b|_\alpha \neq |0|_\alpha$. Then by Lemma 4.3 there exist $U, V \in \alpha$ such that $a \in U$ and $b \in V$. Condition $U, V \in \alpha \in T$ implies by **(tm2)** that UR^tV, which together with $a \in U$ and $b \in V$ implies by Lemma 3.2 that aC^tb.

(iii) ($\Rightarrow$). Suppose $a\mathcal{B}b$. By Lemma 3.2 $a\mathcal{B}b$ implies that there exist ultrafilters U, V such that $U \prec V$, $a \in U$ and $b \in V$. Define $\alpha = \{U\}$ and $\beta = \{V\}$, and by reflexivity of R^t and **(tm2)** we obtain $\alpha, \beta \in T$. Since $U \prec V$ we get $\alpha \prec \beta$. By Lemma 4.3 $a \in U \in \alpha$ and $b \in V \in \beta$ is equivalent to $|a|_\alpha \neq |0|_\alpha$ and $|b|_\beta \neq |0|_\beta$, which together with $\alpha \prec \beta$ implies $h(a)\mathcal{B}h(b)$. Thus $a\mathcal{B}b$ implies $h(a)\mathcal{B}h(b)$.

($\Leftarrow$). Suppose $h(a)\mathcal{B}h(b)$. This implies that there are $\alpha, \beta inT$ such that $\alpha \prec \beta$, $|a|_\alpha \neq |0|_\alpha$ and $|b|_\beta \neq |0|_\beta$. This by Lemma 4.3 implies that there are ultrafilters $U \in \alpha$ and $V \in \beta$ such that $a \in U$ and $b \in V$. Since $\alpha \prec \beta$, this implies by the definition of $\prec$ in T that $U \prec V$. Conditions $a \in U$, $b \in V$ and $U \prec V$ imply by Lemma refultrafilter relations that $a\mathcal{B}b$. Thus $h(a)\mathcal{B}h(b)$ implies $a\mathcal{B}b$.

(iv). In this case we will reason by contraposition: $a \not\leq b$ iff $a.b^* \neq 0$ iff $(a.b^*)C^s(a.b^*)$ iff (by (i)) there exists $\alpha \in T$ such that $(|a|_\alpha.|b|^*_\alpha)C_\alpha(|a|_\alpha.|b|^*_\alpha)$ iff there exists $\alpha \in T$ such that $|a|_\alpha.|b|^*_\alpha \neq |0|_\alpha$ iff there exists $\alpha \in T$ such that $|a|_\alpha \not\leq_\alpha |b|_\alpha$ iff $h(a) \not\leq h(b)$.

(v) The condition follows from the fact that $a \mapsto |a|_\alpha$ is a homomorphism with respect to Boolean operations. □

Appendix E: Proof of Theorem 5.1

Proof. The soundness part of the theorem is easy. For the completeness part we have to show that each consistent set A of formulas has a model. For the proof we will use a kind of canonical model construction. This construction is a variant of the Henkin proof of the completeness theorem for the first-order logic adapted for the logics with additional rules like the rule of transitivity **TR**. This construction is described in [1] Sec. 7 (see also [22] Sec. 3.3), so we refer the reader to consult for the details the above references. The main idea is shortly the following.

Each consistent set A can be extended into a maximal consistent set Γ with some special properties depending on the rules of the logic:

(1) Γ contains all theorems of the logic and is closed under the rule modus ponens,

(2) If the conclusion $\alpha \Rightarrow a\mathcal{B}b$ of the rule **TR** does not belong to Γ then the premise $\alpha \Rightarrow (a\mathcal{B}p \vee p^*\mathcal{B}b)$ also does not belong to Γ for some variable p.

Then, using Γ, one can construct in a canonical way a dynamic contact algebra $\underline{B}$ as follows: define in the set of Boolean terms $a \equiv b$ iff $a = b \in \Gamma$. It can be proved that this is a congruence relation with respect to the Boolean operations which makes possible to define a Boolean algebra over the classes $|a|$ modulo this congruence. We define $|a|C^s|b|$ iff $aC^sb \in \Gamma$ and similarly for the other relations C^t and $\mathcal{B}$. The axioms of dynamic contact algebra guarantee that $\underline{B}$ is a dynamic contact algebra. Moreover the above properties of Γ and the additional axioms and the rule **TR** of the logic guarantee that the obtained dynamic contact algebra satisfies all axioms of the set Ax.

By means of Γ one can define a canonical valuation v in $\underline{B}$ as follows: $v(p) = |p|$ and to prove that $(B, v) \models \alpha$ iff $\alpha \in \Gamma$. Then this shows that (B, v) is a model of Γ and hence a model of A. □